ARL 79

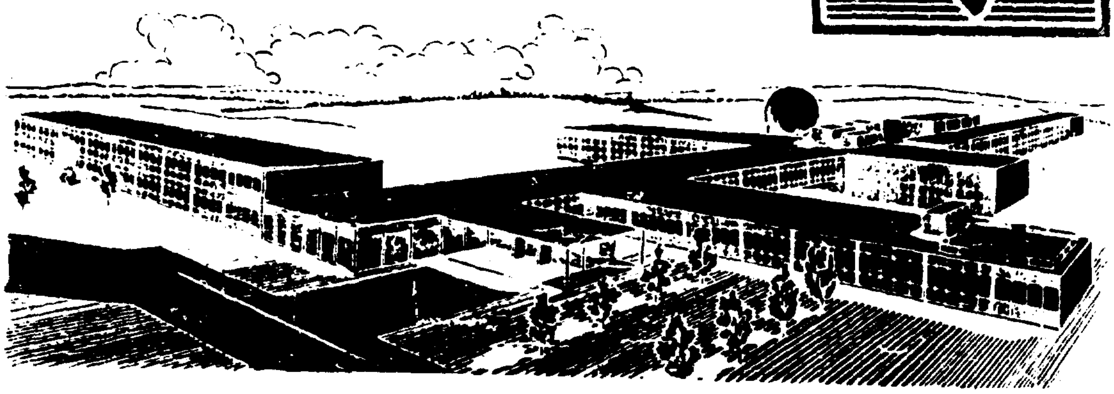
ORTHOGONAL MAIN-EFFECT PLANS

SIDNEY ADDELMAN
OSCAR KEMPTHORNE

IOWA STATE UNIVERSITY
AMES, IOWA

NOVEMBER 1961

AERONAUTICAL RESEARCH LABORATORY
OFFICE OF AEROSPACE RESEARCH
UNITED STATES AIR FORCE



NOTICES

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FOREWORD

This interim technical report was prepared by Iowa State University, Ames Iowa, for the Aeronautical Research Laboratory, Office of Aerospace Research, on Contract AF 33 (616)-5599. The work reported herein was accomplished on Task 70418, "Investigation of Analysis of Variance" of Project 7071, "Mathematical Techniques of Aeromechanics".

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ABSTRACT

This report is concerned with the development and presentation of orthogonal main-effect plans. These plans permit uncorrelated estimates of all main effects of both symmetrical and asymmetrical factorial experiments with a minimum number of trials.

Chapters II and III outline some background material on fitting linear models and factorial experiments which the user of this report may find informative. These two chapters give a short review of existing knowledge of factorial experiments and methods of analysing them.

Chapter IV gives an account of the development of orthogonal main-effect plans for symmetrical and asymmetrical factorial experiments. The plans for asymmetrical experiments are based on the proposition that if the levels of a factor occur with the levels of another factor with proportional frequencies then the two factors are orthogonal. The possibilities of blocking these plans, the efficiencies of the estimates, the randomization procedure and the method of analysis are discussed.

The report concludes with a catalogue of orthogonal main-effect plans. This catalogue consists of the treatment combinations of twenty-six basic plans, involving factors with up to nine levels and with up to eighty-one trials, from which all orthogonal main-effect plans which can be constructed with eighty-one or fewer trials may be deduced.

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I. INTRODUCTION

A. Preliminary Remarks

The purpose of this report is to present in as simple form as possible a catalogue of plans by which the effects of a number of controllable variables can be explored.

The general situation is that there are response or resultant variables or outputs which are thought to depend on controllable variables or inputs, as, for instance, the response of a chemical process, which is a resultant variable, depends on temperature of reaction, pressure, type of catalyst present, flow rate of ingredients and so on. The situation is one of very general occurrence as may be seen from the following examples from widely different areas of human investigation.

Situation	Response	Inputs
1.	Performance of college freshmen students in mathematics	Number of lectures, method of presentation, number of assignments
2.	Performance of astronauts	Types and qualities of foods, other possible environment stimuli, amount of training
3.	Conversion of one chemical to another	Temperature, pressure, feed rates, catalysts, contact time

Situation	Response	Inputs
4.	Quality of an electrical device	Variables in method of production of device, such as nature of alloy, of resistances, rate of cooling in production of parts
5.	Growth of a biological organism	Amounts and types of various nutrients
6.	Yield of an agricultural crop	Rate of seeding, spacing of plants, amounts of fertilizers
7.	Psychological status of sick individuals	Amounts of drugs, amount and nature of psychoanalysis
8.	Treatment of an illness	Diet factors, drug factors, amount of rest
9.	Degree of delinquency of humans	Social and economic measures

The reason for making a list like the above which could be extended indefinitely is to show the range of situations which have the same essential structure. From the point of view of designing an investigation in any of these situations the problems are as follows:

(i) Defining in operational terms the resultant or response variables of interest. In the above examples only one response variable is given, but one can easily imagine many others.

(ii) Defining the control variables, or inputs which should be considered. For various reasons statisticians referred to these inputs as factors and this led to the term factorial experiments, which are nothing but experiments designed to investigate several factors or inputs.

(iii) Defining the variants of factors to be considered, as for instance temperature at 150° C, 200° C, 250° C, or catalyst as manganese or platinum oxide. It is fairly standard to use the term "levels" for these variants. In the case of a factor like temperature which can be envisaged as taking any value in a particular range, that is, a continuous factor, the term "level" is clearly appropriate. For the case of discrete factor, or one in which the variants cannot be represented as points on a line, the term "levels" is not as appropriate because one variant cannot be said to be at a higher level than another. But the use of the term "levels" for both cases does not appear to be confusing to scientists and technologists, and is so well entrenched in statistical terminology, that it will be followed here.

(iv) Specification of the class of situations to which the inputs are to be applied and about which conclusions are desired.

(v) Choice of combinations of the inputs to be tested.

(vi) Assignment of the individual combinations to the members of the class of situations.

(vii) Specification of methods of interpreting the resultant data.

All the above are shorthand statements of problems each of which could be given extensive consideration. Some of the problems are strictly in the province of the experimenter. The statistician per se does not know what yield variables are of interest, what possible inputs should be considered, how the yield variables are to be measured, how the inputs are to be controlled, what levels of the inputs are to be considered, and what class of situations is of interest to the experimenter. The statistician can sometimes give advice on these matters, based on considerations of repeatability of ability to control variables, of precision and sensitivity of possible choices, variability of members of the class of situations to be investigated, and precision of measurement of yield variables. To evaluate whether a statistician can give help on these matters, he will ask questions of the experimenter pointing out the consequences of various conditions and choices and may on the basis of the answers make suggestions or merely content himself with the opinion that the experimenter has already considered these aspects adequately.

It is when we turn to the latter problems of the above list that particular knowledge of statistical design of experiments comes into play. One may for instance envisage a situation in which there are say 10 inputs or factors, each of which could be examined at several levels. The naive reaction is to say: "Try all possible combinations", but when one realizes that even if the number of levels chosen for each factor is 3 the total number of possible combinations is 3^{10} or 59,049, one sees that this is completely impossible. Also it may not be necessary. To see this we need to consider further the type of problem being attacked. That problems can be classified is, we imagine, self-evident, but possibly one of the real gains from statistical thinking is the existence of a classification, which can of course be only rough.

It will be convenient in what follows to use sometimes the phrase "factor space". This is a short term for the totality of possible combinations of the factors or inputs which is considered relevant. If for instance one wishes to investigate temperature between 150° C and 300° C, pressure between 10 lbs/sq. in. and 20 lbs/sq. in., flow rate from 100 gallons per minute to 500 gallons per minute with no restrictions as to what combinations of temperature, pressure and flow rate are possible, the factor space can be represented as a rectangular parallelepiped in 3 dimensions with perpendicular axes representing the three factors. The interior of this parallelepiped then contains a representation of all possible combinations of factors.

Even when the response variables or output variables, and the control or input variables have been defined, problems can be classified according to the end result desired;

(A) The aim may be to determine the combination or combinations of input variables which gives maximum response or minimum response. This is entirely obvious in the case of a chemical process in industry, or in the training of a skilled operator. For brevity this problem is referred to as the problem of response optimization. There may be several response variables which are to be optimized jointly and one may then get into problems of linear or non-linear programming as well.

(B) The aim may be to determine which of the possible factors, which can be imagined as possibly affecting response, do indeed have a non-trivial effect. If one has a production line involving many distinct stages which is producing articles which are not acceptable, one has the problem of determining which of the possible variables, of which there can easily be 20 or more, are affecting the quality of the end product. This problem will be referred to as the problem of screening of factors.

(C) The aim may be to obtain a rough idea of the effects of factors applied jointly over a range of conditions. This problem has not had a particular name associated with it, and for lack of a better term, we shall call it factorial evaluation, that is, evaluation of the role of the possible factors.

(D) The aim may be to obtain a continuous functional relationship of the response (output) to the factors (inputs) like, for instance,

$$y = 5 + \frac{10}{x_1} + 7\frac{x_1}{x_2}$$

where y is the response, and x_1, x_2 are the inputs. This problem will be referred to as functional evaluation.

It is not our purpose here to discuss all these problems in detail, but the following remarks indicate to some extent what is involved and what general plan of investigation should be followed.

In the case of response optimization, it is not essential to obtain much of an idea of what factors are relevant, provided one has included in one's list all the factors which are controllable. One can merely do what might be termed local experimentation in the factor space, by experimenting around a point in the factor space which is thought to be the best guess of the optimal combination. On the basis of this experimentation one finds the direction in the factor space along which it seems best to proceed in search of the optimum. The formulation of a strategy by which to do all this is a difficult problem, but some statisticians have in recent years put forward some interesting ideas. It is of course apparent that the more factors considered the greater the complexity of assessing the experimental results. If there should be two or more response criteria, the problem of optimum seeking becomes much more complicated. For instance one might wish to determine the combination of inputs which gives maximum percentage response of a chemical process subject to the restriction that one wishes to attain a purity greater than a certain percentage, with as low a use of catalyst as possible. Some of the problems which occur on first thought may not even be sufficiently well-defined to enable even the theoretical search for a solution. They may also depend on obtaining a fairly good idea of the functional structure of the situation, that is, the mathematical relationships of the responses or outputs to the inputs.

In the case of the experiment for screening factors, the problem is more one of determining factors which are having appreciable effects with a view to more precise experimentation directed to aims (A), (C) or (D). For instance, suppose that a certain step in a production process consists of heating the uncompleted product entering that stage in a furnace for two hours at 150° C. One might ask whether it matters whether the heating is done for a much shorter time like half an hour or a longer time like - hours. In other words is this a factor which merits some detailed examination or can one assume that realistic changes in the factor level are going to have an inappreciable effect on the resultant product. A procedure commonly used for this sort of investigation is to use the following plan in which there are 6 factors and L denotes a low level, H a high level (or in the case of a discrete factor, two interesting possibilities):

	Factor					
Trial	1	2	3	4	5	6
1	L	L	L	L	L	L
2	H	L	L	L	L	L
3	L	H	L	L	L	L
4	L	L	H	L	L	L
5	L	L	L	H	L	L
6	L	L	L	L	H	L
7	L	L	L	L	L	H

It is not at all difficult to make a convincing case that this plan is very inefficient. The problem is to evaluate 6 factors and the later parts of this technical report exhibit plans which can be shown to have greatest efficiency and sensitivity in determining whether factors merit further study. These are the main-effect plans for which a catalogue is given. The development and cataloging of these plans were the main objectives of the present research.

When we turn to what we have termed "factor evaluation", we are interested in not only what factors have effects of non-trivial importance, but whether also the effect of one factor depends on the status of other factors at which this effect is determined. In standard statistical jargon the question is "What are the effects and interactions of the factors?". For this sort of task, the gamut of factorial experimentation as developed over the past 30 years is relevant.

Finally when we consider the evaluation of functional relationships we not only want to know what factors and interactions are present but we want to express the relationship in as scientifically meaningful way as possible and we have to take account of the units in which factor levels are measured, and have to search for underlying variables which may be composites of the variables on which we choose to experiment. For instance we may experiment on a variable v which is, say, a velocity, but the way velocity enters into the determination of response is in terms of $(v + b)^{1/2}$.

There are common elements to all these aims and there are no sharp divisions among them. In many cases finding the optimum is the ultimate aim, but screening of factors and looking for the possible existence of interactions is undertaken first. Similarly screening of factors and evaluation of interactions may well precede the search for a functional relationship. So the approach to a problem of science or technology is a matter of judgment. An aim of the theoretical study of design of experiments is to construct a rationale to aid the reaching of such a judgment.

B. General Background of Material Presented

The aim of the research underlying this report is to present a catalogue of plans which will enable the experimenter to screen factors. The plans enable the estimation of the effects of all the factors included. Any such estimation is unbiased if there are no interactions. If there are interactions estimates obtained by a model assuming absence of interactions will be deviate from their true values by other than experimental error. This should not be regarded as a deficiency of the plans because the essence of research is the obtaining of ideas which are subjected to confirmation. To demand that an experiment have a completely unambiguous interpretation is realistic only if the experiment will not be repeated, that is, if it is a terminal one, and such experiments must be rare. No decisions in research are irreversible, and knowledge possessed at a particular point of time is at best an approximation to the truth and at worst completely fallacious. Questions underlying this statement can easily be formulated, and one may question, for instance, the risks involved in any plan of investigation.

In addition to the non-terminal nature of research conclusions, one must also take into account what might be termed the economy of research. One can envisage using, at a particular stage of an experimental investigation, a range of plans from the smallest and least-time-consuming plan which will enable one to get some ideas, to a large expensive plan which will give clear-cut unambiguous answers. With the former there is the risk of reaching erroneous conclusions, but the advantage of getting a rough picture quickly. With the latter the risk of reaching erroneous conclusions will be low, but the chance of reaching conclusions which are highly uninteresting may be quite appreciable. Also if the experiment is to take, say, 3 months to perform, one may well find that the ideas which led to its being planned have been modified by experience and knowledge acquired since the planning, so that the "big" experiment only partially done is clearly inappropriate and misdirected. In the case of technological experiments in industry there is obviously a value to be gained from approximate conclusions obtained quickly. Even in what might be termed pure research of no conceivable economic or social value, the researcher will be concerned about the utilization of his own time and energy. It is apparent that one should commit oneself to a large experiment which is seeking a detailed picture only after one has identified factors or inputs which are known to have interesting effects and interactions.

The catalogue is then a catalogue of experimental plans which are likely to be useful in exploratory research. The adjective "exploratory" here is not meant to imply research based on little knowledge but research perhaps in an area which is highly developed, where one wishes to obtain a quick idea of which factors whould be investigated more deeply and which factors should be ignored. There are of course risks involved in ignoring a factor or in deciding that variation in levels of that factor is not worth including in the investigation. It may be that the factor has an interesting effect in only a small range, as, for example, a biological stimulus such as an estrogen. For example, it was known for years that stilbestrol caused some species of animals to have increased growth rates, but it was found that with doses which were thought to have any possible effect the side effects were intolerable. Later it was found that doses which were small relative to doses previously tried had the desired effects with none of the undesired ones.

There is some further insurance of uncertain value in the use of these plans, which arises from the empirical conclusion that there are not likely to be sizeable interactions if there are no main effects. This does emphasize that one should, by one way or another, have some check on the magnitude of error in the situation being examined, because the determination of whether there are effects of interesting magnitude depends on two things (i) whether the actual numerical magnitude is interesting and (ii) whether the actual magnitude is sizeable, say of the order of $1\frac{1}{2}$ or 2 times its standard error.

The catalogue of plans enables an experimenter to discover quickly what plans are available for his particular situation. He may for instance wish to look at two factors at five levels, three factors at four levels, two factors at three levels and one factor at two levels. In the technical language common to the area of the design of experiments, he is involved in a $5^2 \times 4^3 \times 3^2 \times 2$ factorial situation. To list all possible plans would be an impossible task and we have confined ourselves to plans which require no more than 81 observations. The plans listed are orthogonal ones, that is, they enable best unbiased estimates of effects of all factors which are uncorrelated. Even to set out all the possibilities in this case would be tedious but some condensation of the listing is accomplished by giving an index with instructions, so that plans can be used with minor modifications for other situations.

One modification of standard plans which is always possible has been little used in the past. This modification was used in the construction of the plans and can be used to a wider extent. If we have a situation like a $5^2 \times 4^3 \times 3^2 \times 2$ mentioned above we can use a plan for a 5^8 experiment and replace three five-level factors by four-level factors, two five-level factors by three-level factors, and one five-level factor by a two-level factor. In the last case one would set up the following correspondence:

level of five-level factor	0	1	2	3	4
level of two-level factor	0	1	1	1	0

Thus levels 0 and 4 of the five-level factor are replaced by the 0 level of the two-level factor and levels 1, 2 and 3 of the five-level factor by the 1 level of the two-level factor. If one really wanted to experiment with some six-level factors one could collapse a seven-level factor plan. This results in a little loss of statistical efficiency, but not enough to worry about. At least it seems preferable, to the present authors, to encounter a small loss in efficiency in order to accommodate the six-level factor rather than to force the experimenter to delete one of the levels he likes or otherwise revamp the situation. Of course there is no point in introducing levels merely for the sake of doing so, and the more levels that are included for a particular factor, the more trials are required.

C. Structure of the Material Presented

The analysis of the orthogonal main-effect plans, i.e. estimation of parameters, estimation of error, tests of significance, is the standard one based on the method of least squares and a brief account of the features of this method is given in Chapter II.

The basis for most of the plans is the concept of factorial experimentation and the elementary ideas of this topic are presented in Chapter III. The notions of confounding and fractional replication which are essential in the logical development are also presented. In order to present factorial experiments in which the factors have a number of levels equal to the power of a prime number some elementary concepts of Galois field theory are discussed.

In Chapter IV the origin and structure of the plans given in the catalogue are presented. The efficiency of the plans is described and possibilities of blocking are discussed.

The construction of the basic plans presented in the catalogue is described in Chapter V. Several examples of orthogonal main-effect plans constructed from the basic plans are given and an index of the plans which can be obtained from the catalogue presented. The catalogue of basic orthogonal main-effect plans then conclude the report.

D. Notes on Terminology

We give below a short list of terms which occur in the presentation with some explanation of their meaning.

(i) A <u>Factor</u> designates a particular force which is varied in the total investigation at the will and under the control of the experimenter. A factor is also called an input variable or a controlled variable.

(ii) A Quantitative Factor is one whose values can be arranged in order of magnitude. Such values can usually be associated with points on a numerical scale, e.g. temperatures or pressures. This type of factor is also called a continuous factor in the literature.

(iii) A Qualitative Factor is one whose values are not usually arranged in order of magnitude, e.g. type of dosage, batches of material. Although the values of many qualitative factors can be ordered according to a particular criterion they cannot usually be associated with points on a numerical scale.

(iv) Levels are the various values at which a factor is examined, e.g. the levels of temperature in an investigation may be $0^\circ C$, $50^\circ C$, $100^\circ C$ and $150^\circ C$.

(v) A Treatment Combination is one of the possible combinations of levels of all factors under investigation.

(vi) An Experimental Unit is that entity on which a treatment is applied. In experimentation on mice, a single mouse may be the unit. In agronomic investigations the unit is frequently a plot of land. In experimentation on a chemical process the unit could be the system for a prechosen interval of time.

(vii) A Trial is the application of one treatment combination on one experimental unit.

(viii) A Response is the result of a trial with regard to a particular attribute, this result usually being expressed numerically. The response may be the yield of a process, the performance of a machine, the resistance of a material and so on. Usually there will be several response variables for each trial.

(ix) An Experiment is the performance of a planned set of trials.

(x) A Plan is a set of treatment combinations.

(xi) The Effects of a factor are measures of the change in response produced by a change in the level of the factor. When a factor is examined at two levels only, the effect is the difference between the average response of all trials performed at the first level of the factor and that of all trials at the second level. If there are more than two levels the differences between average responses can be expressed in several ways e.g. linear effects, quadratic effects.

(xii) Error is the variability of response in a set of repetitions. It usually consists of components of different origins, e.g. failure of units to be identical, failure to reproduce treatment combinations exactly, inaccuracies of measurement of responses.

II. FITTING LINEAR MODELS OR REGRESSION ANALYSIS

The basis of most parametric analyses of experiments is closely related to the theory of fitting linear models and is frequently referred to as multiple regression*. Regression analysis can be defined as the estimation or prediction of the value of one variable from the values of other given variables.

The assumption in regression analysis is that a variable y may be expressed as a linear function of some known variables $x_1, x_2, \ldots, x_p$ (which may be functionally related) with uncorrelated random deviations which are distributed around zero with constant variance σ^2. This linear function may be expressed as

$$y_\alpha = \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_p x_p + e_\alpha$$

where $x_1, x_2, \ldots, x_p$ take on a particular known value of each α, say, $x_{\alpha 1}, x_{\alpha 2}, \ldots, x_{\alpha p}$. Frequently $x_{\alpha 1} = 1$ for all α.

The best linear unbiased estimate of the β's is obtained by minimizing the sum of squares of deviations

$$\sum_\alpha (y_\alpha - \beta_1 x_{\alpha 1} - \beta_2 x_{\alpha 2} - \ldots - \beta_p x_{\alpha p})^2$$

---

*The term regression was originally introduced to describe, partially, the relationship of one random variable, the dependent variable, to another random variable, the independent variable. In contexts for which regression analysis is widely used, the independent variables are not random variables, so the term is not entirely appropriate.

This procedure is known as the method of least squares. Differentiating the sum of squares of deviations with respect to each of the β's in succession the following equations are obtained:

$$\beta_1 \Sigma x_{\alpha 1}^2 + \beta_2 \Sigma x_{\alpha 1} x_{\alpha 2} + \dots + \beta_p \Sigma x_{\alpha 1} x_{\alpha p} = \Sigma y_\alpha x_{\alpha 1}$$

$$\beta_1 \Sigma x_{\alpha 1} x_{\alpha 2} + \beta_2 \Sigma x_{\alpha 2}^2 + \dots + \beta_p \Sigma x_{\alpha 2} x_{\alpha p} = \Sigma y_\alpha x_{\alpha 2}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\beta_1 \Sigma x_{\alpha 1} x_{\alpha p} + \beta_2 \Sigma x_{\alpha 2} x_{\alpha p} + \dots + \beta_p \Sigma x_{\alpha p}^2 = \Sigma y_\alpha x_{\alpha p}$$

These equations are known as the normal equations. If, as is generally the case in regression problems, the x_i's are not such that one or more linear functions of them are zero, then a unique solution of the above set of p simultaneous equations exists. In order to solve them, first solve p sets of p equations the first set of which is written as follows, using $S_{ij} = S_{ji}$ as an abbreviation for $\sum_\alpha x_{\alpha i} x_{\alpha j}$

$$C_1 S_{11} + C_2 S_{12} + \dots + C_p S_{1p} = 1$$

$$C_1 S_{12} + C_2 S_{22} + \dots + C_p S_{2p} = 0$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$C_1 S_{1p} + C_2 S_{2p} + \dots + C_p S_{pp} = 0$$

Denote the solutions of these equations by $C_{11}, C_{12}, \dots, C_{1p}$, the first subscript indicating that this is the solution for the first set of equations and the second subscript denoting the particular C solution. Next solve these equations with unity on the right-hand side of the second equation and zero on the right-hand side of all the other equations, the solution being denoted by $C_{21}, C_{22}, \ldots, C_{2p}$. Similarly solve the equations with unity at the right-hand side of the third equation, the fourth equation, and so on to the p^{th} equation, in each case the right-hand side of all other equations being zero.

The solutions can be arranged in a $p \times p$ square as follows:

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1p} \\ C_{21} & C_{22} & \cdots & C_{2p} \\ \cdot & \cdot & \cdot & \cdot \\ C_{p1} & C_{p2} & \cdots & C_{pp} \end{bmatrix}$$

This arrangement of the solutions is known as a matrix and is the inverse of the matrix with S_{ij} in place of C_{ij}. The solutions for $\beta_1, \beta_2, \ldots, \beta_p$ are found to be $\hat{\beta}_i = \sum_j C_{ij} P_j$ where $P_j = \sum_\alpha y_\alpha x_{\alpha j}$.

It will be noted that the C_{ij}'s are derived entirely from the x_{ij}'s; that is, they are a function of the structure of the observational setup and are not related to the y's or to the e's. The quantities estimating the β's are linear functions of the y variables. The expectations of the $\hat{\beta}$'s are easily found to be the corresponding β's, the variances of $\hat{\beta}_i$ to be $C_{ii}\sigma^2$ and the covariance of any two $\hat{\beta}$'s, say $\hat{\beta}_i$ and $\hat{\beta}_j$ to be $C_{ij}\sigma^2$. An estimate of σ^2 is derived from the sum of squares of deviations about the estimated values and is given by

$$\hat{\sigma}^2 = s^2 = \frac{1}{n-p} \sum_\alpha (y_\alpha - \hat{\beta}_1 x_{\alpha 1} - \hat{\beta}_2 x_{\alpha 2} - \cdots - \hat{\beta}_p x_{\alpha p})^2$$

$$= \frac{1}{n-p} \left(\sum_\alpha y_\alpha^2 - \sum_i \hat{\beta}_i P_i\right)$$

where $\Sigma \hat{\beta}_i P_i$ is the sum of squares removed by the regression on $x_1, x_2, \ldots, x_p$. The results may be expressed in terms of the analysis of variance, as shown in Table 1.

TABLE 1

ANALYSIS OF VARIANCE

Source	d. f.	Sum of Squares	Mean Square
Regression	p	$\sum_{i=1}^{p} \hat{\beta}_i P_i$	$\Sigma \hat{\beta}_i P_i / p = s_p^2$
Remainder	(n-p)	Difference	Difference/n-p = s^2
Total	n	$\sum_{\alpha=1}^{n} y_\alpha^2$	

In order to test the significance of the regression coefficients (the β_i's) the random deviations e_α are assumed to be normally and independently distributed about a zero mean with constant variance σ^2. With these assumptions the significance of the regression coefficients can be tested jointly by evaluating the mean squares in the analysis of variance and comparing the ratio s_p^2/s^2 to the F distribution with p and (n-p) degrees of freedom.

With the extended assumptions on the random deviations e_α one can also construct confidence intervals for the estimates of each β_i. Since the estimated variance of $\hat{\beta}_i$ is $C_{ii} s^2$ then $(\hat{\beta}_i - \hat{\beta}) / s\sqrt{C_{ii}}$ is distributed as Student's t distribution with (n-p) degrees of freedom.

Hence the 95% confidence intervals on $\hat{\beta}_i$ are given by

$$\hat{\beta}_i \pm t_{n-p,\, 95\%}\; s\sqrt{C_{ii}} \;.$$

Suppose we rename the regression coefficients $\beta_1, \beta_2, \ldots, \beta_q, \beta_{q+1}, \ldots, \beta_p$ and we wish to test whether $\beta_{q+1}, \beta_{q+2}, \ldots, \beta_p$ could be zero making no assumptions about the remaining coefficients.

The procedure is as follows:

(i) Estimate the regression coefficients in the model

$$y_\alpha = \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_p x_p + e$$

obtaining $\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_p$. The sum of squares removed by the the regression on $x_1, x_2, \ldots, x_p$ is equal to $\sum_{i=1}^{p} \hat{\beta}_i P_i$.

(ii) Estimate the regression coefficients in the model

$$y_\alpha = \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_q x_q + e$$

obtaining $\beta_1^*, \beta_2^*, \ldots, \beta_q^*$. The sum of squares removed by the regression on $x_1, x_2, \ldots, x_q$ is equal to $\sum_{i=1}^{q} \beta_i^* P_i$.

(iii) Construct the analysis of variance given in Table 2.

TABLE 2

ANALYSIS OF VARIANCE

Source	d. f.	Sum of Squares	Mean Square
Regression on $x_1, \ldots, x_q$	q	$\sum_{i=1}^{q} \beta_i^* P_i$	s_q^2
Regression on $x_{q+1}, \ldots, x_p$ after fitting $x_1, \ldots, x_q$	p-q	$\sum_{i=1}^{p} \hat{\beta}_i P_i - \sum_{i=1}^{q} \beta_i^* P_i$	s_d^2
Regression on $x_1, \ldots, x_p$	p	$\sum_{i=1}^{p} \hat{\beta}_i P_i$	s_p^2
Remainder	n-p	Difference	s^2
Total	n	$\sum_{\alpha=1}^{n} y_\alpha^2$	

To test the hypothesis that $\beta_{q+1}, \ldots, \beta_p$ are zero we utilize the fact that under the hypothesis that they are zero the ratio s_d^2/s^2 will be distributed as F with $(p-q)$ and $(n-p)$ degrees of freedom and thus compare s_d^2/s^2 with the value in the F table corresponding to $(p-q)$ and $(n-p)$ degrees of freedom.

The usual regression test devised to test whether deviations about the mean have a regression on the independent x variates may be deduced from the above discussion. The complete hypothesis is that

$$y_\alpha = \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_p x_p + e_\alpha$$

and the restricted hypothesis that

$$y_\alpha = \beta_1 x_{\alpha 1} + e$$

where $x_{\alpha 1}$ is unity for all values of α. The estimate β_1^* is $\bar{y}$ and the sum of squares due to the regression on x_1 (i.e. the sum of squares due to the mean) is $\bar{y}\Sigma y$. The "correction for the mean" $\bar{y}\Sigma y$, with one degree of freedom may be deducted from the total sum of squares and the analysis is given in Table 3.

TABLE 3

ANALYSIS OF VARIANCE

Source	d.f.	Sum of Squares	Mean Square
Regression on $x_2, \ldots, x_p$	p-1	$\sum_{i=2}^{p} \hat{\beta}_i P_{ic}$	s_r^2
Remainder	n-p	Difference	s^2
Total	n-1	$\sum_{\alpha=1}^{n} y_\alpha^2 - \bar{y} \sum_{\alpha=1}^{n} y_\alpha$	

$P_{ic} = P_i - \bar{x}_{i1} \Sigma y_\alpha$ denotes the sum of products around the mean.

Much of the preceeding discussion can be simplified through the use of matrix notation. The linear function expressing y as a function of the x variates may be written as

$$y = X\beta + e$$

where

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_\alpha \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}, \quad X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ \cdot & \cdot & \cdot & \cdot \\ x_{\alpha 1} & x_{\alpha 2} & \cdots & x_{\alpha p} \\ \cdot & \cdot & \cdot & \cdot \\ x_{n1} & x_{n2} & \cdots & x_{np} \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \beta_p \end{bmatrix} \text{ and } \quad e = \begin{bmatrix} e_1 \\ \cdot \\ \cdot \\ \cdot \\ e_\alpha \\ \cdot \\ \cdot \\ \cdot \\ e_n \end{bmatrix}$$

The sum of squares to be minimized is

$$e'e = (y - X\beta)'(y - X\beta) = y'y - 2\beta'X'y + \beta'X'X\beta .$$

The normal equations are

$$S\hat{\beta} = X'y$$

where $S = X'X$.

If S is non-singular then $\hat{\beta} = S^{-1}X'y$ and the variance-covariance matrix of the estimates is equal to $\sigma^2 S^{-1}$. The estimate of σ^2 is given by

$$\hat{\sigma}^2 = (y - X\hat{\beta})'(y - X\hat{\beta})/(n-p) = (y'y - \hat{\beta}'X'y)/(n-p) .$$

The results presented above will be utilized in the chapters which follow and matrix notation will be used whenever it simplifies the presentation.

There arises the problem of how should σ^2 be estimated if $n = p$ (i. e. the number of parameters to be estimated equals the number of trials). The estimate of σ^2 based upon the sum of squares of deviations about the estimated values of the parameters is sometimes called pure error. If $n = p$ it is clear from the formula for $\hat{\sigma}^2$ that no estimate of pure error can be derived from the experiment. In such a situation there are two possible ways of resolving the problem. First, the experimenter may be investigating a process for which the experimental error is already known. In this case the error obtained from prior information may be used as an estimate of σ^2. This estimate of σ^2 can then be based on infinite degrees of freedom and the estimation and test of significance procedure can be made as if the estimate of experimental error had been obtained from the experiment itself.

A graphical procedure for analysing factorial experiments developed by Daniel (1959) may be useful in obtaining a rough estimate of error. This procedure uses a half-normal grid on which to plot the absolute values of the contrasts defining the main effects and interactions of a factorial experiment. If these contrasts are arranged in order of absolute magnitude and plotted on half-normal probability paper they should fall along a straight line, if all factors have no effects.

A half-normal grid can be prepared by taking a sheet of arithmetic (normal) probability paper, deleting the printed probability scale P, for the range $P < 50\%$ and replacing each value of $P > 50\%$ by the corresponding value of $P' = 2P - 100$. The relation

$$P' = (i - \frac{1}{2})/N \,;\; i = 1,\ 2,\ \ldots,\ N,$$

where N is the number of main effects and interactions to be estimated, is used for plotting the empirical distribution of contrasts. The abscissae are the absolute values of the contrasts.

Under the null hypothesis that all factors have no effects the standard error of each contrast σ_c, could be roughly estimated by the contrast for which P' is most nearly 0.683. If it is known that some effects or interactions are likely to be real and it appears from the graph that they are, then they should be judged real and the remaining contrasts used to determine the standard error i. e. reduce N by the number of real effects and/or interactions in the formula $P' = (i - \frac{1}{2})/N$ and then estimate the standard error of a contrast by the absolute value of the contrast for which the new P' is most nearly 0.683. If a straight line is drawn through the origin and the absolute value of the contrast for which P' is most nearly 0.683 a rough idea of which effects and interactions are significantly large can be obtained. These will fall far to the right of the line. An estimate of the experimental error, σ^2, can be obtained from the formula

$$\hat{\sigma}^2 = \hat{\sigma}_c^2/(N+1) .$$

The estimate of experimental error is based upon N degrees of freedom and although it is approximate and deduced by subjective reasoning, it does give some information about the experiment that would not be forthcoming without an estimate of error. The reader who is interested in this technique can find many illustrative examples of its use in the paper by Daniel (1959).

III. FACTORIAL EXPERIMENTS

When an experiment involves several factors, the effects of all factors on a characteristic of interest may be investigated simultaneously by varying each factor so that all or a suitable subset of all possible combinations of the factors are considered. An experiment in which this procedure is used is known as a factorial experiment.

A. Factorial Experiments with Factors at Two Levels

The simplest and most common factorial experiments involve factors which occur at two levels. The two levels of a factor, may be denoted by 0 and 1. A treatment is denoted by a particular combination of levels, one level from each factor. The treatment combination for which all the factors occur at the 0 level can be simply denoted by (1). The 1 level of a factor, say factor A, can also be represented by the lower case letter a. A factorial experiment involving three factors A, B and C each at two levels would consist of the following treatment combinations: (1), a, b, ab, c, ac, bc and abc. In these combinations the presence of a letter indicates that the corresponding factor occurs at its 1 level and the absence of a letter indicates the corresponding factor occurs at its 0 level.

The main effect of factor A is defined to be the difference between the mean of the yields at the 1 level of factor A and the mean of the yields at the 0 level of factor A.

Hence the main effect of A is $\frac{1}{4}(a + ab + ac + abc) - \frac{1}{4}((1) + b + c + bc)$ which can also be written as

$$A = \frac{1}{4}(a-1)(b+1)(c+1)$$

where the expression is to be expanded algebraically and the responses substituted for the treatment symbols. The effects and interactions of the 2^3 factorial experiment are given by

$$A = \frac{1}{4}(a-1)(b+1)(c+1)$$

$$B = \frac{1}{4}(a+1)(b-1)(c+1)$$

$$AB = \frac{1}{4}(a-1)(b-1)(c+1)$$

$$C = \frac{1}{4}(a+1)(b+1)(c-1)$$

$$AC = \frac{1}{4}(a-1)(b+1)(c-1)$$

$$BC = \frac{1}{4}(a+1)(b-1)(c-1)$$

$$ABC = \frac{1}{4}(a-1)(b-1)(c-1)$$

a minus sign appearing in any factor on the right if the letter is present on the left. We will adhere to the convention that treatment combinations are represented by lower-case letters and effects and interactions by capitals.

It will be noted that the effects and interactions are seven mutually orthogonal contrasts of the responses of the eight treatment combinations.

	(1)	a	b	ab	c	ac	bc	abc
4 A	-	+	-	+	-	+	-	+
4 B	-	-	+	+	-	-	+	+
4 AB	+	-	-	+	+	-	-	+
4 C	-	-	-	-	+	+	+	+
4 AC	+	-	+	-	-	+	-	+
4 BC	+	+	-	-	-	-	+	+
4 ABC	-	+	+	-	+	-	-	+

Orthogonality of two linear contrasts may be defined as follows: Consider two linear functions, C_1 and C_2, of the variates $x_1, x_2, \ldots, x_n$ where the x's have the same variance and are uncorrelated.

$$C_1 = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$$

$$C_2 = \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_n x_n$$

where α_i and β_i may assume any values, not all zero. A necessary and sufficient condition that the two linear functions be orthogonal is

$$\sum_{i=1}^{n} \alpha_i \beta_i = 0$$

If the mean response of the eight treatment combinations is denoted by μ the effects and interactions are represented by

$$
\begin{bmatrix} 8 & \mu \\ 4 & A \\ 4 & B \\ 4 & AB \\ 4 & C \\ 4 & AC \\ 4 & BC \\ 4 & ABC \end{bmatrix}
=
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
-1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
-1 & 1 & 1 & -1 & 1 & -1 & -1 & 1
\end{bmatrix}
\begin{bmatrix} (1) \\ a \\ b \\ ab \\ c \\ ac \\ bc \\ abc \end{bmatrix}
$$

With n factors A, B, C, D, etc. the effects and interactions may be represented by

$$X = \frac{1}{2^{n-1}} (a \pm 1)(b \pm 1)(c \pm 1)(d \pm 1) \ldots$$

where the sign in each bracket is positive if the corresponding capital letter is not contained in X and negative if it is contained in X, and the whole expression or the right-hand side is to be expanded algebraically and the yields substituted in place of the corresponding treatment combinations.

The choice of the divisor in the above expression is a matter of convention only and depends upon the definition of an effect or interaction. Here we have defined an effect or interaction on the basis of the difference between two experimental units.

The response of a treatment combination may be written as $a_i\, b_j\, c_k \ldots$ where absence is denoted by the subscript taking the value zero and presence by the subscript taking the value unity. Then

$$a_i b_j c_k \ldots = \mu \pm \tfrac{1}{2}A \pm \tfrac{1}{2}B \pm \tfrac{1}{2}AB \pm \tfrac{1}{2}C \pm \tfrac{1}{2}AC \pm \ldots$$

where the sign on $\frac{1}{2}A$ is - if $i = 0$ and + if $i = 1$

on $\frac{1}{2}B$ is - if $j = 0$ and + if $j = 1$

on $\frac{1}{2}C$ is - if $k = 0$ and + if $k = 1$

and so on,

and the sign on a term involving several letters is the product of the signs on the individual letters.

If a 2^n experiment is replicated r times in randomized blocks of 2^n plots each effect or interaction is estimated by the mean of $r\,2^{n-1}$ responses minus the mean of $r\,2^{n-1}$ responses and therefore has a variance of $\left(\frac{1}{r\,2^{n-1}} + \frac{1}{r\,2^{n-1}}\right)\sigma^2 = \sigma^2/r\,2^{n-2}$. Furthermore the estimates of effects and interactions are uncorrelated so that the variance of any linear function of them can be easily obtained.

B. Factorial Experiments with Factors at More than Two Levels

The 3^n system

With factors at three levels the effect of any one factor may be expressed in several ways. First the response at each level, where the level is represented by 0, 1 or 2, can be expressed as a deviation from the mean response at the three levels, giving say A_0, A_1 and A_2 where $A_0 + A_1 + A_2 = 0$. Another approach is that the main effect of a factor can be represented by independent comparisons among the means corresponding to the different levels of the factors. Among three independent quantities there are two independent comparisons. The comparisons which are of interest will depend upon the nature of the factors, in particular whether they are qualitative or quantitative.

If the levels are qualitative and the 0 level denotes the control and the other two denote treatments the comparisons of interest may be (i) a comparison of the two treatments and (ii) a comparison of the average of the two treatments with the control. These comparisons can be expressed as

$$A' = \frac{1}{2}(2a_0 - a_1 - a_2)$$

$$A'' = a_1 - a_2$$

respectively, where a_0, a_1 and a_2 denote the responses with factor A at the 0, 1 and 2 levels.

For most quantitative factors the comparisons of interest will be those giving the most information on the relation between the responses and the levels, namely the slope and the curvature. This can be represented by a polynomial expression $y = \alpha_0 + \alpha_1 x + \alpha_2 x^2$, where x denotes the levels of the factor and y is the response variable. The linear and quadratic effects of factor A may be written as:

$$A_L = (a_2 - a_0)$$

$$A_Q = (a_2 - 2a_1 + a_0) \ .$$

The quadratic effect is the linear contrast among a_0, a_1 and a_2 which is orthogonal to the linear effect.

Now consider two quantitative factors A and B, each at three equally spaced levels. The interaction of these two factors will be the interaction of a 3×3 table and will have four degrees of freedom.

These four degrees of freedom may be separated into orthogonal contrasts each with a single degree of freedom.

$$A_L B_L = (a_2 - a_0)(b_2 - b_0)$$

$$A_Q B_L = (a_2 - 2a_1 + a_0)(b_2 - b_0)$$

$$A_L B_Q = (a_2 - a_0)(b_2 - 2b_1 + b_0)$$

$$A_Q B_Q = (a_2 - 2a_1 + a_0)(b_2 - 2b_1 + b_0)$$

This system of expressing the results may be extended indefinitely.

Several conventions have been used to define the main effects and interactions, each convention having some merit. One common convention adopted is to define the effects and interactions on the basis of the difference between two experimental units. Adopting this convention the main effects and interactions of an experiment on two three-level factors A and B, are given by

$$A_L = \frac{1}{3}(a_2 - a_0)(b_0 + b_1 + b_2)$$

$$A_Q = \frac{1}{6}(a_0 - 2a_1 + a_2)(b_0 + b_1 + b_2)$$

$$B_L = \frac{1}{3}(a_0 + a_1 + a_2)(b_2 - b_0)$$

$$B_Q = \frac{1}{6}(a_0 + a_1 + a_2)(b_0 - 2b_1 + b_2)$$

$$A_L B_L = \frac{1}{2}(a_2 - a_0)(b_2 - b_0)$$

$$A_L B_Q = \frac{1}{4}(a_2 - a_0)(b_0 - 2b_1 + b_2)$$

$$A_Q B_L = \frac{1}{4}(a_0 - 2a_1 + a_2)(b_2 - b_0)$$

$$A_Q B_Q = \frac{1}{8}(a_0 - 2a_1 + a_2)(b_0 - 2b_1 + b_2)$$

The convention adopted does not alter any tests of significance performed on the parameters and therefore need be of little concern to the experimenter.

There is a class of experiments involving quantitative and qualitative factors in which the treatment combinations have an appearance of consisting of a full set of factorial combinations but are not in fact so. A simple example of this type is that in which there are three equally spaced amounts, including a zero amount of a particular treatment administered by three methods. Since the zero amounts of the treatment administered by the three methods are identical treatments there are only seven different treatment combinations and not nine. The experimenter must consider whether he should use the nine treatment combinations as though they were all distinct or only the seven distinct combinations, and further he should consider the method of analysis in each case. For a more detailed discussion of this type of experiment the reader is referred to section 18.8 of Kempthorne (1952).

We now present a formal method of defining effects and interactions. Consider the case of three factors A, B and C each at two levels 0 and 1. The eight treatment combinations (1), a, b, ab, c, ac, bc, abc may be represented by the points (0,0,0), (1,0,0), (0,1,0), (1,1,0), (0,0,1), (1,0,1), (0,1,1) and (1,1,1) respectively, in Euclidean space with axes x_1, x_2 and x_3, the first coordinate referring to the level of factor A, the second to the level of factor B and the third to the level of factor C. The effects and interactions defined previously have a simple algebraic interpretation. The effect of A is the comparison of the treatment combinations for which $x_1 = 0$ with those for which $x_1 = 1$.

Likewise the effect of B is the comparison of the treatment combinations for which $x_2 = 0$ with those for which $x_2 = 1$ and the effect of C is the comparison of the treatment combinations for which $x_3 = 0$ with those for which $x_3 = 1$. The interaction AB, for example, is in the former notation the comparison among treatment combinations,

$$(1) + c + ab + abc - a - b - ac - bc$$

i.e. of the points $(0,0,0)$, $(0,0,1)$, $(1,1,0)$ and $(1,1,1)$ versus the points $(1,0,0)$, $(0,1,0)$, $(1,0,1)$ and $(0,1,1)$. For the points $(0,0,0)$ and $(0,0,1)$, $x_1 + x_2 = 0$ and for the points $(1,1,0)$ and $(1,1,1)$, $x_1 + x_2 = 2$ and for the other four points, $x_1 + x_2 = 1$. If the numbers are reduced modulo 2, that is, any number is replaced by the remainder when it is divided by 2, the interaction is the comparison of those treatment combinations for which $x_1 + x_2 = 0 \pmod 2$ versus those for which $x_1 + x_2 = 1 \pmod 2$. It is easily verified that the effects and interactions are based on a comparison of two groups of treatment combinations given by the equations in Table 4.

TABLE 4

EQUATIONS REPRESENTING EFFECTS AND INTERACTIONS

Effect or Interaction	Left-Hand Side of Equation
A	x_1
B	x_2
AB	$x_1 + x_2$
C	x_3
AC	$x_1 + x_3$
BC	$x_2 + x_3$
ABC	$x_1 + x_2 + x_3$

For example the treatment combinations entering ABC with a minus sign are (1), ab, ac and bc and for these $x_1 + x_2 + x_3 = 0 \pmod 2$ and the treatment combinations entering with a plus sign are a, b, c, and abc for which $x_1 + x_2 + x_3 = 1 \pmod 2$.

The above approach for the 2^n system suggests the appropriate approach for the 3^n system. Consider the arrangement of the nine treatment combinations with two factors each at three levels.

	Level of factor A $\rightarrow x_1$		
Level of factor B $\downarrow x_2$	(0, 0)	(1, 0)	(2, 0)
	(0, 1)	(1, 1)	(2, 1)
	(0, 2)	(1, 2)	(2, 2)

The main effect of factor A can be represented by the comparisons among three means: those for which $x_1 = 0$, for which $x_1 = 1$ and for which $x_1 = 2$. A representation of these effects as two linearly independent numbers may be obtained by considering each mean as a deviation from the over-all mean. The interaction of factors A and B has four degrees of freedom. These four degrees of freedom can be considered from the point of view of the completely orthogonalized 3×3 square:

$$\begin{array}{ccc} A_\alpha & B_\beta & C_\gamma \\ B_\gamma & C_\alpha & A_\beta \\ C_\beta & A_\gamma & B_\alpha \end{array}$$

The comparisons among the columns give the effect of factor A, and among the rows the effect of factor B. Those among the Latin letters and those among the Greek letters each with two degrees of freedom represent the four degrees of freedom for the interaction of the two factors. Consider the following grouping given by the Latin letters: (0, 0), (2, 1), (1, 2) versus (1, 0), (0, 1), (2, 2) versus (2, 0), (0, 2), (1, 1). For this grouping the comparisons are among those treatment combinations for which $x_1 + x_2 = 0, = 1, = 2 \pmod 3$. Similarly the comparisons among the Greek letters are comparisons among the treatment combinations for which $x_1 + 2x_2 = 0, = 1, = 2 \pmod 3$.

The pair of degrees of freedom corresponding to the equations $x_1 + x_2 = 0, = 1, = 2$ may be denoted by the symbol AB and the pair corresponding to $x_1 + 2x_2 = 0, =1, =2$ by AB^2. The interaction degrees of freedom may also be represented by BA and BA^2 respectively. It is easily verified that the comparisons among the groups of treatment combinations represented by BA and BA^2 are the same as those represented by AB and AB^2 respectively. It is necessary, in order to obtain a complete and unique enumeration of the pairs of degrees of freedom, to adopt the rule that an order of the letters is to be chosen in advance and that the power of the first letter in a symbol must be unity. If the power of the first letter of a symbol is 2 then by squaring the symbol and using the rule that any letter cubed is to be replaced by unity the power of the first letter will be unity. This process may be extended indefinitely. For three factors the results are shown in Table 5.

The extensions are quite straightforward and need not be enumerated. For the 3^n system there are n independent factors and their generalized interactions, giving rise to $(3^n - 1)/2$ symbols each representing two degrees of freedom.

The symbols used above to denote pairs of degrees of freedom can also be used to denote the magnitudes of effects and interactions. Each symbol represents a comparison among three groups of 3^{n-1} treatment combinations, examples of which are:

TABLE 5

EQUATIONS REPRESENTING EFFECTS AND INTERACTIONS

Effect or Interaction	Left-Hand Side of Equation
A	x_1
B	x_2
AB	$x_1 + x_2$
AB^2	$x_1 + 2x_2$
C	x_3
AC	$x_1 + x_3$
AC^2	$x_1 + 2x_3$
BC	$x_2 + x_3$
BC^2	$x_2 + 2x_3$
ABC	$x_1 + x_2 + x_3$
ABC^2	$x_1 + x_2 + 2x_3$
AB^2C	$x_1 + 2x_2 + x_3$
AB^2C^2	$x_1 + 2x_2 + 2x_3$

A_0 = (mean of treatment combinations for which $x_1 = 0 \pmod 3$) - (mean of all treatment combinations)

AB_0 = (mean of treatment combinations for which $x_1 + x_2 = 0 \pmod 3$) - (mean of all treatment combinations)

AB^2_1 = (mean of treatment combinations for which $x_1 + 2x_2 = 1 \pmod 3$) - (mean of all treatment combinations)

AB^2C_2 = (mean of treatment combinations for which $x_1 + 2x_2 + x_3 = 2$ (mod 3)) - (mean of all treatment combinations).

With these definitions the response of treatment combination $a_i\, b_j\, c_k$ in terms of effects and interactions is

$$a_i b_j c_k = \mu + A_i + B_j + AB_{i+j} + AB^2_{i+2j} + C_k + AC_{i+k} + AC^2_{i+2k} + BC_{j+k} + BC^2_{j+2k} + ABC_{i+j+k} + ABC^2_{i+j+2k} + AB^2C_{i+2j+k} + AB^2C^2_{i+2j+2k}$$

where all subscripts are reduced modulo 3 and μ is the mean of all combinations. For example, the response of treatment combination $a_1\, b_0\, c_2$ is given by

$$a_1 b_0 c_2 = \mu + A_1 + B_0 + AB_1 + AB^2_1 + C_2 + AC_0 + AC^2_2 + BC_2 + BC^2_1 + ABC_0 + ABC^2_2 + AB^2C_0 + AB^2C^2_2 .$$

Thus it is possible to express any linear contrast of the responses in terms of the effects and interactions.

Now suppose that the treatment combinations are tested the same number of times in a randomized block trial. Then, with an additive model, the observed response will be equal to a true response plus an error. The errors may be regarded as uncorrelated with mean zero and constant variance σ^2. Then the best estimate of any contrast of the true responses is the same contrast of the observed means.

The only estimable functions of the parameters are functions of the type $\alpha_i - \alpha_j$ where α is one of the set of symbols A, B, AB, AB^2, ABC etc. and the i, j have values equal to 0, 1 or 2. It is easily verified that the estimates of quantities $\alpha_i - \alpha_j$ and $\beta_m - \beta_n$, where α, β are different ones of the set of symbols are uncorrelated.

Consider the nine treatment combinations of the 3^2 factorial experiment written in terms of effects and interactions.

$$
\begin{aligned}
a_0 b_0 &= \mu + A_0 + B_0 + AB_0 + AB_0^2 \\
a_0 b_1 &= \mu + A_0 + B_1 + AB_1 + AB_2^2 \\
a_0 b_2 &= \mu + A_0 + B_2 + AB_2 + AB_1^2 \\
a_1 b_0 &= \mu + A_1 + B_0 + AB_1 + AB_1^2 \\
a_1 b_1 &= \mu + A_1 + B_1 + AB_2 + AB_0^2 \\
a_1 b_2 &= \mu + A_1 + B_2 + AB_0 + AB_2^2 \\
a_2 b_0 &= \mu + A_2 + B_0 + AB_2 + AB_2^2 \\
a_2 b_1 &= \mu + A_2 + B_1 + AB_0 + AB_1^2 \\
a_2 b_2 &= \mu + A_2 + B_2 + AB_1 + AB_0^2
\end{aligned}
$$

The estimate of $A_2 - A_0$, say, is clearly equal to a constant times

$$(a_2 b_0 + a_2 b_1 + a_2 b_2 - a_0 b_0 - a_0 b_1 - a_0 b_2)$$

and the estimate of $B_2 - B_1$ is equal to a constant times

$$(a_0 b_2 + a_1 b_2 + a_2 b_2 - a_0 b_1 - a_1 b_1 - a_2 b_1)$$

The coefficients of the treatments for the above two contrasts are, apart from the constant multiplier

	a_0b_0	a_0b_1	a_0b_2	a_1b_0	a_1b_1	a_1b_2	a_2b_0	a_2b_1	a_2b_2
$A_2 - A_0$	-1	-1	-1	0	0	0	1	1	1
$B_2 - B_1$	0	-1	1	0	-1	1	0	-1	1

Since the sum of the products of corresponding coefficients is zero the two contrasts are orthogonal.

Among the three deviations from the overall mean A_0, A_1 and A_2, say, there are two independent contrasts. These can be represented by the contrasts $A_2 - A_0$ and $A_2 - 2A_1 + A_0$. If the same types of contrasts are utilized for each of the other symbols, it is possible to obtain eight orthogonal contrasts.

Since in each of the two systems of defining effects and interactions the treatment combinations can be written in terms of the effects and interactions it is not difficult to determine the relationship of one ystem to another. For example, in the 3^2 experiment on factors A and B

$$A_L = a_2 b_0 + a_2 b_1 + a_2 b_2 - a_0 b_0 - a_0 b_1 - a_0 b_2$$

and $$A_2 - A_0 = a_2 b_0 + a_2 b_1 + a_2 b_2 - a_0 b_0 - a_0 b_1 - a_0 b_2$$

Thus $A_L = A_2 - A_0$. Similarly, it can be shown that $A_Q = A_0 - 2A_1 + A_2$. Now, $A_L B_L = a_2 b_2 + a_0 b_0 - a_0 b_2 - a_2 b_0$. If these four treatment combinations are substituted in the equation

$$a_i b_j = \mu + A_i + B_j + AB_{i+j} + AB^2_{i+2j}$$

it is easily demonstrated that

$$A_L B_L = AB_0 + AB_1 - 2AB_2 + 2AB^2_0 + AB^2_1 - AB^2_2$$

The p^n system

The following presentation is a straightforward generalization of the 2^n and 3^n systems. The generalization from the 3^n system to the p^n system, where p is a prime number, can be seen fairly easily, without introducing proofs. The proofs are based on the properties of Galois fields which will be given later.

Represent the treatment combination by numbers $x_1 x_2 \ldots x_n$, where x_i is the level of the i^{th} factor in the particular combination. The numbers x_i take on values from 0 to $(p-1)$. All the numbers are reduced modulo p, that is, a number greater than $(p-1)$ is replaced by the remainder after division by p. The (p^n-1) degrees of freedom among the p^n treatment combinations may be partitioned into $(p^n-1)/(p-1)$ sets of $(p-1)$ degrees of freedom. Each set of $(p-1)$ degrees of freedom is given by the contrasts among the p sets of p^{n-1} treatment combinations specified by the following p equations:

$$\left.\begin{array}{l} \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 0 \\ \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 1 \\ \ldots\ldots\ldots\ldots\ldots \\ \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = (p-1) \end{array}\right\} \pmod{p}$$

The α_i's must be positive integers between 0 and $(p-1)$, not all equal to zero and for uniqueness the coefficient of the first α_i that is not zero equals unity.

Two sets of $(p-1)$ degrees of freedom resulting from equations with left-hand sides $\Sigma\alpha_i x_i$ and $\Sigma\beta_i x_i$ will be orthogonal unless $\beta_i = k\alpha_i$ for each i. This can easily be seen because the two equations,

$$\begin{aligned} \Sigma \alpha_i x_i &= k \\ \Sigma \beta_i x_i &= m \end{aligned} \qquad (\text{mod } p)$$

will be satisfied by p^{n-2} treatment combinations, if β_i is not equal to a constant multiplier of α_i.

The symbol $A^{\alpha_1} B^{\alpha_2} \ldots K^{\alpha_n}$, which corresponds to the equations whose left-hand side is

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n ,$$

denotes a set of $(p-1)$ degrees of freedom, the power of the first letter occurring being restricted to be unity.

Galois field theory

In order to obtain a procedure for investigating factors, each having s levels, where $s = p^m$, a knowledge of group theory is essential.

A set of s elements $u_0, u_1, \ldots, u_{s-1}$ is said to be a finite field of order s if the following properties hold:

(i) The set is closed under addition and multiplication, i.e. if u_i and u_j belong to the set then so do $u_i + u_j$ and $u_i u_j$.

(ii) Addition and multiplication are commutative, i.e.

$u_i + u_j = u_j + u_i$ and $u_i u_j = u_j u_i$.

(iii) Addition and multiplication are associative, i.e.

$u_i + (u_j + u_k) = (u_i + u_j) + u_k$ and $(u_i u_j) u_k = u_i (u_j u_k)$

(iv) The distributive law holds, i.e.

$u_i (u_j + u_k) = u_i u_j + u_i u_k$

(v) There exists an identity element u_0, under addition, i. e.

$u_0 + u_j = u_j$ for any j .

(vi) There exists an identity element u_1, under multiplication, i. e.

$u_1 u_j = u_j$ for any j .

(vii) For each element u_i there exists a unique inverse with respect to addition, i. e.

$u_i + u_{i'} = u_0$.

(viii) For each element u_i $(\neq u_0)$ there exists a unique inverse with respect to multiplication, i. e.

$u_i u_{i'} = u_1$.

The finite field of p elements, where p is a prime number may be represented by $u_0 = 0$, $u_1 = 1$, $u_2 = 2 \ldots u_{p-1} = p-1$, in which addition and multiplication are the ordinary arithmetic operations with the rule that the numbers are to be reduced modulo p .

In general, a Galois field of p^m elements is obtained as follows: Let $P(x)$ be a given polynomial in x of degree m with integral coefficients; and let $F(x)$ be any polynomial in x with integral coefficients. Then $F(x)$ may be expressed as

$$F(x) = f(x) + p.q(x) + P(x)\,Q(x)$$

here $q(x)$ and $Q(x)$ may be any polynomial in x with integral coefficients and

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{m-1} x^{m-1}$$

and the coefficients $a_0, a_1, \ldots, a_{m-1}$ belong to the set $0, 1, 2, \ldots, p-1$. This relationship may be written as

$$F(x) = f(x) \mod p,\ P(x)$$

and $f(x)$ is said to be the residue of $F(x)$ modulis p and $P(x)$. If p and $P(x)$ are kept fixed then the $f(x)$'s form p^m classes of functions. It may be readily verified that when p is a prime number and $P(x)$ is irreducible modulo p, that is, $P(x)$ cannot be expressed in the form

$$P(x) = P_1(x)\, P_2(x) + p \cdot P_3(x)$$

then the classes defined by the $f(x)$'s make up a field.

The finite field formed by the p^m classes of residues is called a Galois field of order p^m and is denoted by $GF(p^m)$. The p^m classes are the same, regardless of the choice of $P(x)$, subject to the restrictions imposed above, and the field $GF(p^m)$, always exists if p is a prime and m a positive integer. The classes of residues can be represented by the different possible functions $f(x)$ and may also be denoted by $u_0, u_1, u_2, \ldots, u_{s-1}$ where $s = p^m$.

To illustrate, we shall obtain the Galois field of 3^2 elements. An irreducible polynomial modulo 3 is $P(x) = 1 + x^2$. Now consider the possible functions $f(x)$. These are of the form $a_0 + a_1 x$ where a_0 and a_1 are elements of the set 0, 1 and 2. Hence the elements of the field are: $u_0 = 0$, $u_1 = 1$, $u_2 = 2$, $u_3 = x$, $u_4 = 2x$, $u_5 = 1 + x$, $u_6 = 1 + 2x$, $u_7 = 2 + x$, $u_8 = 2 + 2x$. There is a further theorem that all the elements or marks of the field except the zero element u_0 can be represented as the powers of an element known as a primitive mark. It is readily verified that $y = 1 + x$ is a primitive mark. For

$$y = 1 + x$$

$$y^2 = 1 + 2x + x^2 = 2x \quad (\text{since } P(x) = 1 + x^2 = 0)$$

$$y^3 = 2x + 2x^2 = 2x + 1 + 2(1 + x^2) = 1 + 2x$$

$$y^4 = 4x^2 = x^2 = 2 + 1 + x^2 = 2$$

$$y^5 = 2 + 2x$$

$$y^6 = 2 + 2x + 2x + 2x^2 = 4x + 2(1 + x^2) = 4x = x$$

$$y^7 = x + x^2 = x + 2 + (1 + x^2) = 2 + x$$

$$y^8 = 4 = 1$$

Both the representations of the elements of the field are important in that the representation in terms of x is used for addition and the representation in terms of y for multiplication.

An irreducible polynomial $P(x)$, a primitive mark and the addition and multiplication tables for $GF(2^2)$, $GF(2^3)$ and $GF(3^2)$ are now presented.

TABLE 6

THE GALOIS FIELD, $GF(2^2)$

$P(x) = 1 + x + x^2$, Primitive mark = x.

Addition

	0	1	2	3
0	0	1	2	3
1		0	3	2
2			0	1
3				0

Multiplication

	0	1	2	3
0	0	0	0	0
1		1	2	3
2			3	1
3				2

TABLE 7

THE GALOIS FIELD, $GF(2^3)$

$P(x) = 1 + x^2 + x^3$, Primitive mark = x

Addition

	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1		0	3	2	5	4	7	6
2			0	1	6	7	4	5
3				0	7	6	5	4
4					0	1	2	3
5						0	3	2
6							0	1
7								0

Multiplication

	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1		1	2	3	4	5	6	7
2			4	6	5	7	1	3
3				5	1	2	7	4
4					7	3	2	6
5						6	4	1
6							3	5
7								2

TABLE 8

THE GALOIS FIELD, $GF(3^2)$

$P(x) = 1 + x^2$, Primitive mark = $1 + x$

Addition

	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1		2	0	4	5	3	7	8	6
2			1	5	3	4	8	6	7
3				6	7	8	0	1	2
4					8	6	1	2	0
5						7	2	0	1
6							3	4	5
7								5	3
8									4

Multiplication

	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1		1	2	3	4	5	6	7	8
2			1	6	8	7	3	5	4
3				2	5	8	1	4	7
4					6	1	7	2	3
5						3	4	6	2
6							2	8	5
7								3	1
8									6

The use of these fields in examining the $s^n = (p^m)^n$ factorial system is exactly analogous to the use of $0, 1, 2, \ldots, p-1$ for the p^n factorial system. The treatment combinations may be denoted by $(x_1, x_2, \ldots, x_n)$ where each x_i can take one of the values $0, 1, 2, \ldots, s-1$. The $(s^n-1)/(s-1)$ sets of $(s-1)$ degrees of freedom, which can be obtained by partitioning the (s^n-1) degrees of freedom into main effects and interactions, are orthogonal, and the responses may be expressed in terms of the mean, effects and interactions. The only complication is that the numbers used are marks of the Galois field, addition and multiplication being defined within the field.

C. Confounding in Factorial Experiments

The performance of a comparative experiment requires definition of experimental units and the precision of conclusions depends on the variation among the units, in addition to other things. The greater the variation among units the higher the error and the lower the precision. To combat this, it is advantageous to group the units into what are usually called blocks of units and to design the experiment so that only the variation among units within blocks enters the standard error of estimates. The smaller the block size the more uniform the units in the block will tend to be. It is therefore desirable to have some means of reducing the size of the block, i.e. the number of units in each block, and thus increase precision. For this purpose the device of confounding has been found very useful.

Consider a simple situation of three factors, A, B and C each at two levels, the eff cts and interactions being defined as in Table 9 (apart from the conventional numerical divisor).

Suppose that the eight treatment combinations are arranged in two blocks according to their sign in the ABC interaction. The two blocks would then contain the following treatment combinations:

Block 1	Block 2
(1)	a
ab	b
ac	c
bc	abc

TABLE 9

EFFECTS AND INTERACTIONS OF THE 2^3 EXPERIMENT

	(1)	a	b	ab	c	ac	bc	abc
A	-1	1	-1	1	-1	1	-1	1
B	-1	-1	1	1	-1	-1	1	1
AB	1	-1	-1	1	1	-1	-1	1
C	-1	-1	-1	-1	1	1	1	1
AC	1	-1	1	-1	-1	1	-1	1
BC	1	1	-1	-1	-1	-1	1	1
ABC	-1	1	1	-1	1	-1	-1	1

The quantity used to estimate A is orthogonal to blocks in that it is given by $\frac{1}{4}(-(1) + a - b + ab - c + ac - bc + abc)$ and of the four treatment combinations entering the estimate positively two are in each block, and likewise for the four treatment combinations entering negatively. The estimate will then contain none of the additive block effects. The same is true of the other main-effects and the two-factor interactions.

The three-factor interaction is estimated by $\frac{1}{4}(-(1) + a + b - ab + c - ac - bc + abc)$ and this estimate measures not only the true ABC interaction but also the block difference (block 2 minus block 1). It is not possible to separate the true interaction from the block difference and the interaction and block difference are said to be completely confounded with each other. Thus the three-factor interaction cannot be estimated. In many situations it is known that the high order interactions are trivial and therefore can be used as blocking factors.

The set of treatment combinations in the block of a confounded experiment which includes the control treatment is called the intrablock subgroup.

If none of the interactions can be considered trivial and smaller blocks are desired, the experiment can be replicated several times with a different effect or interaction confounded with blocks in each replicate. For example, in the 2^3 experiment we might replicate the experiment four times confounding ABC with blocks in the first replicate, AB with blocks in the second replicate, AC with blocks in the third replicate and BC with blocks in the fourth replicate. Thus each interaction may be estimated in the three replicates in which it is unconfounded. This type of confounding is known as partial confounding.

The rule of the generalized interaction for 2^n experiments is that if effects or interactions represented by X and Y are confounded, then so is XY, obtained by multiplying the symbols together equating any letter which is squared to unity. The rule of the generalized interaction for the 3^n system is that if pairs of degrees of freedom corresponding to X and Y are completely confounded, then so are the pairs of degrees of freedom corresponding to XY and XY^2 where any letter cubed is equated to unity. By adopting the rule that in any symbol the power of the first letter should be unity a complete and unique specification of all effects and interactions is achieved.

An example of the use of this symbolism is the following. Suppose a

3^3 experiment is to be arranged in blocks of three. In any system of confounding four pairs of degrees of freedom must be confounded with blocks. For example, if AB^2C and AC^2 are confounded, so is

$$AB^2C \times AC^2 = A^2B^2C^3 = A^2B^2 = A^4B^4 = AB$$

and

$$AB^2C \times AC^2AC^2 = A^3B^2C^5 = B^2C^2 = B^4C^4 = BC\ .$$

The composition of the blocks is easily obtained from the definition of the effects and interactions. In the above example there are nine blocks given by the solutions of the equations

$$x_1 + 2x_2 + x_3 = i \pmod 3$$

$$x_1 + 2x_3 = j \pmod 3$$

where i and j each take on the values 0, 1 and 2.

The rule of the generalized interaction for the p^n experiment is that, if effects or interactions denoted by X and Y are completely confounded with blocks, then so are the $(p-1)$ sets of $(p-1)$ degrees of freedom denoted by XY, XY^2, ..., XY^{p-1}, where any letter raised to the p^{th} power is to be replaced by unity and the resultant symbol is to be raised to such power as makes the first letter in it have a power of unity. This may be proved as follows: Let X correspond to the equations

$$\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = 0, = 1, = \dots = (p-1) \pmod p$$

and Y to the equations

$$\beta_1 x_1 + \beta_2 x_2 + \dots + \beta_n x_n = 0, = 1, = \dots = (p-1) \pmod p\ .$$

Because X and Y are confounded completely with blocks, the treatment combinations of any one block satisfy the equations

$$\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = i \pmod p$$

$$\beta_1 x_1 + \beta_2 x_2 + \dots + \beta_n x_n = j \pmod p$$

where i and j are each one of the numbers 0, 1, ..., (p-1). For these treatment combinations the equations may be combined to give

$$(\alpha_1 + \lambda\beta_1) x_1 + (\alpha_2 + \lambda\beta_2) x_2 + \dots + (\alpha_n + \lambda\beta_n) x_n = i + \lambda j \pmod p$$

where λ can take on any value from 1 to (p-1) and the coefficients on both sides of the equation must be reduced modulo p. This equation corresponds to the symbol XY^{λ}. Thus, the treatment combinations of any block take on a constant value for any one of the equations corresponding to XY^{λ} where λ is any value from 1 to (p-1). The effect or interaction XY^{λ} is therefore confounded with blocks for these values of λ.

D. Fractional Replication

A complete factorial experiment investigating all possible combinations of all the levels of the different factors will involve a large number of trials when the number of factors is five or more. When the number of factors is large the number of trials required may even become prohibitive. One is therefore led to consider the economy of space and material which will be attained by using only a fraction of the possible number of treatment combinations at the expense of losing some information inherent in a complete replicate. The general process by which information can be obtained from less than a full replicate of a factorial experiment is known as fractional replication.

Suppose that three factors, A, B and C, each having two levels are under investigation and it is known that these factors do not interact. The relation between the true responses and the effects and interactions can be presented in tabular form as follows:

TABLE 10

RELATION BETWEEN RESPONSES AND EFFECTS AND INTERACTIONS IN A 2^3 EXPERIMENT

	μ	$\frac{1}{2}A$	$\frac{1}{2}B$	$\frac{1}{2}AB$	$\frac{1}{2}C$	$\frac{1}{2}AC$	$\frac{1}{2}BC$	$\frac{1}{2}ABC$
(1)	+	-	-	+	-	+	+	-
a	+	+	-	-	-	-	+	+
b	+	-	+	-	-	+	-	+
ab	+	+	+	+	-	-	-	-
c	+	-	-	+	+	-	-	+
ac	+	+	-	-	+	+	-	-
bc	+	-	+	-	+	-	+	-
abc	+	+	+	+	+	+	+	+

Suppose that only the four treatment combinations that enter the ABC interaction negatively are considered; namely (1), ab, ac and bc. It is clear from the table that it is impossible to separate the mean μ from the ABC interaction. Similarly the A effect cannot be separated from the BC interaction, the B effect cannot be separated from the AC interaction and the C effect cannot be separated from the AB interaction. The estimating equations for this plan are:

$$\mu - \frac{1}{2}ABC = \frac{1}{4}[(1) + ab + ac + bc]$$

$$\frac{1}{2}(A - BC) = \frac{1}{4}[-(1) + ab + ac - bc]$$

$$\frac{1}{2}(B - AC) = \frac{1}{4}[-(1) + ab - ac + bc]$$

$$\frac{1}{2}(C - AB) = \frac{1}{4}[-(1) - ab + ac + bc]$$

With only the four trials, A is completely confounded or aliased with BC, B with AC, C with AB and μ with ABC.

If the factors do not interact all the interactions may be neglected and the estimating equations can then be used to estimate the mean and the three main effects, and these estimates are uncorrelated.

The treatments chosen for the 1/2 replicate were selected as those which entered the ABC interaction negatively. The selection could have been those which entered the ABC interaction positively. For most purposes, it does not matter which of the two halves of the experiment is chosen.

If by convention, μ is denoted by I, the confounding relation may be expressed as

$$I = ABC .$$

This relation is known as the defining contrast or identity relationship where the equal sign is used to denote "completely confounded with". The remaining three confounding relations may be written as

$$A = BC$$

$$B = AC$$

$$C = AB$$

These relations may be obtained from the identity relationship by multiplying both sides by an effect of interest with the rule that any letter which is squared is to be replaced by unity. Thus $I = ABC$ when multiplied by A gives

$$A = A^2BC = BC .$$

It should be noted that the 1/2 replicate of the 2^3 experiment given by the identity relationship $I = ABC$ consists of the same treatment combinations as one of the blocks of a 2^3 experiment in two blocks of four plots each where ABC is confounded with blocks.

The identity relationship of a quarter replicate of a 2^n experiment is of the form

$$I = X = Y = XY$$

where X, Y and XY are higher order interactions and XY is the generalized interaction of X and Y.

If, for a 1/3 replicate of the 3^3 experiment, the identity relationship is given by

$$I = ABC$$

the following confounding relations may be generated:

$$\begin{aligned}
A &= AB^2C^2 = BC \\
B &= AB^2C \quad = AC \\
C &= ABC^2 \quad = AB \\
AB^2 &= AC^2 \quad = BC^2
\end{aligned}$$

In general a $1/p^m$ replicate of a p^n factorial experiment may be specified by an identity relationship of the form

$$I = X = Y = XY = XY^2 = \ldots = XY^{p-1} = Z = XZ = XZ^2 = \ldots = XZ^{p-1}$$

$$= YZ = YZ^2 = \ldots = YZ^{p-1} = XY^r Z^s \quad [r,\ s = 1,\ 2,\ \ldots,\ (p-1)]$$

$$= \text{etc.}$$

where there are t independent contrasts X, Y, Z, etc.

For a more detailed discussion of factorial experiments one may refer to the texts by Kempthorne (1952) and Davies (1954).

IV. ORTHOGONAL MAIN-EFFECT PLANS

The experimental plans which are developed in this chapter and subsequently presented in the catalogue are called orthogonal main-effect plans as they permit the estimation of all main effects without correlation, when all interactions are negligible.

The most commonly used factorial experiments involve factors which all occur at the same number of levels. These experiments are known as symmetrical factorial experiments. A good deal is already known about the construction of orthogonal main-effect plans for symmetrical factorial experiments although a comprehensive catalogue of such plans has never been published. There are a great many experimental situations which involve factors that do not all occur at the same number of levels. These experiments are known as asymmetrical factorial experiments. Heretofore the standard technique for constructing orthogonal main-effect plans for asymmetrical factorial experiments has been to combine two or more orthogonal main-effect plans for different symmetrical experiments. Hence in order to construct an orthogonal main-effect plan for the $3^4 \times 2^3$ experiment one would combine the plan for the 3^4 experiment in nine trials with the plan for the 2^3 experiment in four trials to obtain the required plan in thirty-six trials. This procedure often requires more trials than the experimenter can afford to make.

The orthogonal main-effect plans developed in this chapter for both symmetrical and asymmetrical factorial experiments require the least number of trials that has yet been attained for such plans. For many experiments the suggested plans are so highly fractionated that there are few if any degrees of freedom available for the estimation of experimental error. In such situations one must use an estimate of experimental error which is (i) known from previous experience, (ii) derived from some of the degrees of freedom available for estimating main-effects which prior knowledge indicates are negligible or (iii) approximated by a procedure which utilizes a graph and is known as the half-normal plot technique of interpreting factorial experiments.

The plans consist of th reatment combinations which permit uncorrelated main effect estimates. The treatment combinations are denoted by the level at which each factor occurs. Thus the treatment combination 0112 in an experiment on four factors is that combination for which the first factor occurs at its first level, the second and third factors occur at their second levels and the third factor occurs at its third level.

A. Weighing Plans

The problem of estimating the weights of small objects placed on a balance scale was first considered by Yates (1935). The weighing problem is concerned with the development of plans for estimating the effects of two-level factors with as few trials as possible. Since it can be assumed that the weight of a set of objects is the sum of the weights of the individual objects, all interactions may be presumed to be absent. Hotelling (1944) constructed optimum (in the sense of minimum variance) plans for estimating the weights of (N-1) objects with N weighings on a chemical balance scale. He proved that a necessary and sufficient condition for attaining an optimum weighing plan is that the design matrix, X say, be a Hadamard matrix, which is a matrix consisting of 1's and -1's such that $X'X$ = diagonal $(N, N, \ldots, N)$ where N is the number of weighings. Paley (1933) proved that a sufficient condition that a Hadamard matrix of size N exist is $N \equiv 0 \pmod 4$, with the exception of $N = 2$ which is a trivial case.

Plackett and Burman (1946) provided what is effectively a complete solution of the weighing problem when the estimates of the weights are required to be uncorrelated. Most of the plans which they developed can be generated by a cyclic shifting of the elements of one treatment combination successively $(N-2)$ times and then adding the control treatment. When the number of trials $N \equiv 0 \pmod 4$ is not of the form $N = 2^n$ the orthogonal main-effect plans given in the catalogue have been generated by cyclically shifting the elements of the treatment combinations presented by Plackett and Burman.

1. Plans for Symmetrical Factorial Experiments

Orthogonal main-effect plans can be constructed easily for symmetrical factorial experiments involving $(s^n-1)/(s-1)$ factors, each having s levels, with s^n treatment combinations, where $s = p^m$ and p is a prime number. The $(s^n-1)/(s-1)$ factors can be represented by n factors each having $s(= p^m)$ levels and all their generalized interactions. Hence one need only choose the treatment combinations from a complete s^n factorial plan and assign one of the $(s^n-1)/(s-1)$ factors to each of the factors and interactions of the s^n plan.

Let the n factors of the s^n factorial plan be represented by

$X_1, X_2, \ldots, X_n$ and their generalized interactions by $k_1 X_1 + k_2 X_2 + \ldots + k_n X_n$ where $k_i (i = 1, 2, \ldots, n)$ can take on any value of the Galois field $GF(p^m)$ and it is understood that the coefficient of the first factor appearing in an interaction is unity. The notation adopted here for the generalized interactions differs from the standard notation for interactions as given, for example by Kempthorne (1952), in order to facilitate the presentation which follows later.

The procedure for constructing orthogonal main-effect plans will be illustrated with a plan for four factors A, B, C and D, each having three levels with nine treatment combinations. In this example $s = 3$, $n = 2$ and $(s^n-1)/(s-1) = 4$. The four factors can be represented by two factors X_1 and X_2 of the 3^2 factorial experiment and their generalized interactions $X_1 + X_2$ and $X_1 + 2X_2$. The treatment combinations which comprise the orthogonal main-effect plan are

0	0	0	0
0	1	1	2
0	2	2	1
1	0	1	1
1	1	2	0
1	2	0	2
2	0	2	2
2	1	0	1
2	2	1	0

The interactions which are members of the defining contrast (identity relationship) may be determined by choosing those interactions whose X representation equals 0 (mod 3). The generators of the interactions in defining contrast for the example given above are ABC^2 and ACD, since the X representation of ABC^2 is $X_1 + X_2 + 2(X_1 + X_2) = 0$ (mod 3) and of $ACD = X_1 + (X_1 + X_2) + (X_1 + 2X_2) = 0$ (mod 3).

Some plans which may be constructed by this method are given in Table 11.

TABLE 11

INDEX OF SOME MAIN-EFFECT PLANS

Number of levels	Maximum Number of factors	Number of observations
2	3	4
2	7	8
2	15	16
2	31	32
2	63	64
3	4	9
3	13	27
3	40	81
4	5	16
4	21	64
5	6	25
7	8	49
8	9	64
9	10	81

The orthogonal main-effect plans with s^n treatment combinations which accommodate up to $(s^n-1)/(s-1)$ factors can be augmented to yield orthogonal main-effect plans with $2s^n$ treatment combinations.

The augmented plans can accommodate up to $[2(s^n-1)/(s-1) - 1]$ factors, each having $s = p^m$ levels. In order to illustrate the theory underlying the augmentation procedure some preliminary lemmas are now developed.

Let $u_0, u_1, \ldots, u_{s-1}$ represent the elements of the Galois field $GF(p^m)$ and let $u_0^2, u_1^2, \ldots, u_{s-1}^2$ represent the squares of the elements of $GF(p^m)$. The set of squared elements of $GF(p^m)$ will be denoted by $GF^2(p^m)$. It is easily verified that apart from the 0 element, the set $GF^2(p^m)$ forms a cyclic Abelian group under multiplication. It follows from the cyclic property that (i) when $p = 2$, $GF^2(p^m)$ contains each of the elements of $GF(p^m)$ and (ii) when p is an odd prime, the elements of $GF^2(p^m)$ comprise a subset of $(s+1)/2$ distinct elements of $GF(p^m)$, where one element occurs once and $(s-1)/2$ elements are duplicated.

Consider one of the factors X_i in a main-effect plan in which each X_i has s levels each occurring s^{n-1} times in a total of s^n treatment combinations. Let X_i^2 be a pseudo-factor obtained by squaring the levels of X_i. The following lemmas can now be presented:

Lemma 1: When p is an odd prime, $X_i^2 + kX_i$ (k an element of $GF(p^m)$) contains $(s+1)/2$ distinct levels, one level occurring s^{n-1} times and $(s-1)/2$ levels occurring $2s^{n-1}$ times in s^n treatment combinations.

Lemma 2: When $p = 2$, X_i^2 contains each of the s levels s^{n-1} times.

Lemma 3: When $p = 2$, $X_i^2 + kX_i$ $(k \neq 0)$ contains $s/2$ distinct levels each occurring $2s^{n-1}$ times.

Lemma 3 can be proved as follows. Let x_i range over the elements of $GF(p^m)$ which represent the s levels of X_i. As x_i ranges over the elements of the field so does $x_i + k$ where k is an element of $GF(p^m)$. Also if $x_i + k = x_j \pmod 2$ then $x_j + k = x_i \pmod 2$. Hence $x_i(x_i + k) = x_i x_j$ and $x_j(x_j + k) = x_i x_j$. Thus whatever values of $x_i(x_i + k)$ are achieved they are achieved for at least two values of x_i.

It will now be shown that the values of $x_i(x_i + k)$ are achieved for exactly two values of x_i. Let y be the generator of the field and let $x_i = y^\alpha$ and $k = y^\beta$. Thus $x_i(x_i + k) = y^\alpha(y^\alpha + y^\beta)$.

Suppose that

$$y^\alpha(y^\alpha + y^\beta) = y^\gamma(y^\gamma + y^\beta)$$

where

$$y^\alpha \neq y^\gamma \text{ and } y^\alpha + y^\beta \neq y^\gamma \quad .$$

Hence

$$(y^\alpha)^2 + y^\alpha y^\beta = (y^\gamma)^2 + y^\gamma y^\beta$$

$$(y^\alpha + y^\gamma)^2 + (y^\alpha + y^\gamma) y^\beta = 0$$

$$(y^\alpha + y^\gamma)(y^\alpha + y^\gamma + y^\beta) = 0 \quad .$$

This implies that either $y^\alpha + y^\gamma = 0$ and therefore $y^\alpha = y^\gamma$ which is a contradiction or that $y^\alpha + y^\gamma + y^\beta = 0$ and therefore $y^\alpha + y^\beta = y^\gamma$ which is a contradiction. Hence the values of $x_i(x_i + k)$ are achieved for exactly two values of x_i and Lemma 3 is proved.

Lemma 4: The factor represented by $X_i^2 + k_i X_i + \sum_{j \neq i} k_j X_j$, where at least one $k_j \neq 0$, contains each of the s levels s^{n-1} times.

Lemma 5: The levels of $X_i^2 + k_1 X_i + k_2 X_j$ which occur in a plan with the u_t level of $a_1 X_i + a_2 X_j$, where k_1, k_2, a_1 and a_2 are elements of $GF(p^m)$ and $a_2 \neq 0$ are given by the values of $x_i^2 + k_1 x_i + k_2 x_j + c(a_1 x_i + a_2 x_j) - cu_t$ where $k_2 + ca_2 = 0$ and x_i ranges over the elements of $GF(p^m)$.

Proof: When $a_1 X_i + a_2 X_j$ takes on the u_t level then $a_1 x_i + a_2 x_j = u_t$ and thus

$$x_j = (u_t - a_1 x_i)/a_2 .$$

Hence the levels of the factor $X_i^2 + k_1 X_i + k_2 X_j$ which occur with the lev·l u_t of $a_1 X_i + a_2 X_j$ can be represented by

$$x_i^2 + k_1 x_i + k_2 x_j = x_i^2 + k_1 x_i + k_2(u_t - a_1 x_i)/a_2$$

$$= x_i^2 + (k_1 - k_2 a_1/a_2)x_i + (k_2/a_2) u_t .$$

Since $k_2 + ca_2 = 0$, then $c = -k_2/a_2$.

Thus

$$x_i^2 + (k_1 - k_2 a_1/a_2) x_i + (k_2/a_2) u_t = x_i^2 + k_1 x_i + k_2 x_j + c(a_1 x_i + a_2 x_j) - cu_t ,$$

and the lemma is proved.

Two factors X_i and X_j are said to be orthogonal to each other if each level of X_j occurs the same number of times with every level of X_i. Two factors X_i and X_j are said to be semi-orthogonal to each other if (i) for p an odd prime, one level of X_j occurs s^{n-2} times and $(s-1)/2$ levels of X_j each occur $2s^{n-2}$ times with each level of X_i and (ii) for $p = 2$, $s/2$ levels of X_j each occur $2s^{n-2}$ times with each level of X_i.

It follows from Lemmas 1, 3, and 5 that when p is an odd prime or when $k_1 - k_2 a_1/a_2 \neq 0$, then $a_1 X_i + a_2 X_j$ is semi-orthogonal to $X_i^2 + k_1 X_i + k_2 X_j$. It follows from Lemmas 2 and 5 that when $p = 2$ and $k_1 - k_2 a_1/a_2 = 0$ then $a_1 X_i + a_2 X_j$ is orthogonal to $X_i^2 + k_1 X_i + k_2 X_j$. Employing an argument similar to that used in Lemma 5 it can be deduced that $kX_i^2 + k_1 X_i + X_j$ and $kX_i^2 + k_2 X_i + X_j$ are orthogonal to each other when $k_1 \neq k_2$.

Lemma 5 can be generalized to include more than two factors as stated in Lemma 5a.

Lemma 5a: The levels of $X_i^2 + k_i X_i + \sum_{j \neq i} k_j X_j$ which occur in a plan with the u_t level of $a_i X_i + \sum_{j \neq i} a_j X_j$ are given by the values of

$$x_i^2 + k_i x_i + \sum_{j \neq i} k_j x_j + c(a_i x_i + \sum_{j \neq i} a_j x_j) - cu_t$$

where $k_j + ca_j = 0$ for all $j \neq i$. If the a_j and the k_j are not of such a form that $k_j + ca_j = 0$ for all $j \neq i$ and some c contained in $GF(p^m)$ then the two factors are orthogonal.

Lemma 6: When p is a prime the complements in $GF(p^m)$ to the elements in $GF^2(p^m)$ are the set of elements in $GF^2(p^m)$ each multiplied by an element of $GF(p^m)$ which is not an element of $GF^2(p^m)$. If the set of elements in $GF^2(p^m)$ and their set of complements are taken together in one set the elements of $GF(p^m)$ are obtained.

Proof: From abstract group theory (see Birkhoff and MacLane (1953)) we employ a lemma which states that two right cosets of a subgroup are either identical or without common elements. Now the elements of

$GF^2(p^m)$ form an Abelian subgroup of the elements of $GF(p^m)$. Hence multiplying each element of $GF^2(p^m)$ by an element of $GF(p^m)$ which is not an element of $GF^2(p^m)$ yields the complementary set to $GF^2(p^m)$.

It is clear from Lemma 2 that when $p = 2$ the set complementary to $GF^2(p^m)$ is the null set.

We can now present

Theorem 1: There exists a main-effect plan for $[2(s^n-1)/(s-1) - 1]$ factors, each at $s = p^m$ levels, with $2s^n$ treatment combinations.

Proof: In order to facilitate the presentation of the proof of Theorem 1, let $n = 2$. First construct an orthogonal main effect plan for $(s^2-1)/(s-1)$ factors each at s levels in s^2 trials, represented by the two factors X_1 and X_2 and their generalized interactions $X_1 + X_2, X_1 + 2X_2, \ldots, X_1 + (s-1)X_2$. To these add $[(s^2-1)/(s-1) - 1]$ factors represented by $X_1^2 + X_2, X_1^2 + X_1 + X_2, X_1^2 + 2X_1 + X_2, \ldots, X_1^2 + (s-1)X_1 + X_2$. These $[2(s^n-1)/(s-1) - 1]$ factors in s^n observations represent the first half of the main-effect plan.

Note from the preceeding lemmas that when p is a prime number, $X_1 + a_i X_2$ and $X_1^2 + k_i X_1 + X_2$ are semi-orthogonal and also that X_2 and $X_1^2 + k_i X_1 + X_2$ are semi-orthogonal for all a_i and k_i in $GF(p^m)$ except $a_i = 0$. All other pairs of factors are clearly orthogonal. If $p = 2$ and $(k_i - a_1/a_i) = 0$, then $a_1 X_1 + a_i X_2$ and $X_1^2 + k_i X_1 + X_2$ are orthogonal.

The second half of the plan is chosen so that the pairs of factors which are orthogonal in the first half are also orthogonal in the second half and pairs of factors which are semi-orthogonal in the first half are semi-orthogonal in a complementary manner in the second half. The factors in the second half which correspond to the factors of the first half can be denoted by

$$X_1, X_2, X_1+X_2+b_1, X_1+2X_2+b_2, \ldots, X_1+(s-1)X_2+b_{s-1}, kX_1^2+X_2,$$
$$kX_1^2+k_1X_1+X_2+c_1, kX_1^2+k_2X_1+X_2+c_2, \ldots, kX_1^2+k_{(s-1)}X_1+X_2+c_{s-1}$$

where the coefficients $b_1, b_2, \ldots, b_{s-1}, k, k_1, k_2, \ldots, k_{s-1}, c_1, c_2, \ldots, c_{s-1}$ are to be determined.

From Lemma 5, it is seen that the levels of $X_1^2+X_2$ which occur with the u_t level of X_2 are given by the values of $x_1^2+u_t$ where x_1 takes on the values of the elements of $GF(p^m)$. Without loss of generality we may let $u_t = u_0 = 0$. When p is an odd prime, the values of $kX_1^2+X_2$, where k is an element of $GF(p^m)$ but not an element of $GF^2(p^m)$, which occur with the $u_t = 0$ level of X_2 are given by the values of kx_1^2. As shown in Lemma 6, kx_1^2 complements x_1^2.

Thus, when p is an odd prime k can take on the value of any element in $GF(p^m)$ which is not an element of $GF^2(p^m)$. If $p = 2$ it is clear from Lemma 2 that $k = 1$.

A method for determining the constants $b_1, b_2, \ldots, b_{s-1}$, $k_1, k_2, \ldots, k_{s-1}$, $c_1, c_2, \ldots, c_{s-1}$, when $s = p^m$ and p is an odd prime is now presented. In order that the levels of $kX_1^2+X_2$ which occur with the 0 level of $X_1+a_iX_2+b_i$ be the complements of the levels of $X_1^2+X_2$ which occur with the 0 levels of $X_1+a_iX_2$, b_i must be such that the values which $kx_1^2-(1/a_i)x_1-b_i/a_i$ takes when x_1 ranges over the field $GF(p^m)$ complements the values which $x_1^2-(1/a_i)x_1$ takes. Now $x_1^2-(1/a_i)x_1$ consists of one element of

$GF(p^m)$ occurring once and $(s-1)/2$ elements occurring twice. Let the unique element of $GF(p^m)$ be u_1. Then $x_1^2 - (1/a_i)\, x_1 = u_1$ must have only one solution as x_1 ranges over the elements of $GF(p^m)$. Thus $1/a_i^2 + 4u_1 = 0$ and hence $4u_1 = -1/a_i^2$. Since $kx_1^2 - (1/a_i)x_1 - b_i/a_i$ must complement $x_1^2 - (1/a_i)x_1$ the equation

$$kx_1^2 - (1/a_i)x_1 - b_i/a_i = u_1$$

must also have only one solution.

Therefore

$$1/a_i^2 + 4k(b_i/a_i + u_1) = 0.$$

Substituting $4u_1 = -1/a_i^2$ in this equation and solving for b_i we get

$$b_i = (k-1)/4ka_i. \qquad (1)$$

To find the levels of $X_1^2 + d_i X_1 + X_2$ which occur with the 0 levels of X_2 note that there exists an element of $GF(p^m)$, u_2 say, such that $x_1^2 + d_i x_1 = u_2$ has only one solution.

Thus $d_i^2 + 4u_2 = 0$ and hence $4u_2 = -d_i^2$. In order that the levels of $kX_1^2 + k_i X_1 + X_2 + c_i$ which occur with the 0 levels of X_2 complement those given by $x_1^2 + d_i x_1$, then $kx_1^2 + k_i x_1 + c_i = u_2$ must have only one solution. Substituting $4u_2 = -d_i^2$ in this equation and solving for c_i we get

$$c_i = k_i^2/4k - d_i^2/4. \qquad (2)$$

To find the levels of $X_1^2 + d_i X_1 + X_2$ which occur with the 0 levels of $X_1 + a_i X_2$ note that there exists an element of $GF(p^m)$, u_3 say, such that $x_1^2 + (d_i - 1/a_i)x_1 = u_3$ has only one solution.

Thus

$$(d_i - 1/a_i)^2 + 4u_3 = 0 \quad \text{and} \quad 4u_3 = -(d_i - 1/a_i)^2 .$$

Since $kx_1^2 + (k_1 - 1/a_i)x_1 + (c_i - b_i/a_i)$ must complement $x_1^2 + (d_i - 1/a_i)x_1$, the equation

$$kx_1^2 + (k_1 - 1/a_i)x_1 + (c_i - b_i/a_i) = u_3$$

must also have only one solution as x_1 ranges over the elements of $GF(p^m)$. Therefore

$$(k_i - 1/a_i)^2 - 4k\left[(c_i - b_i/a_i) - u_3\right] = 0 .$$

Substituting $4u_3 = -(d_i - 1/a_i)^2$ and equations (1) and (2) into this equation we get

$$k_i = kd_i . \tag{3}$$

Hence equation (2) can be rewritten as

$$c_i = d_i^2\,(k-1)/4 . \tag{4}$$

Thus k is determined by choosing an element of $GF(p^m)$ which is not an element of $GF^2(p^m)$. By letting $a_i = 1, 2, \ldots, s-1$ we can determine $b_1, b_2, \ldots, b_{s-1}$ from equation (1). Then setting $d_i = 1, 2, \ldots, s-1$ we determine $k_1, k_2, \ldots, k_{s-1}$ from equation (3) and $c_1, c_2, \ldots, c_{s-1}$ from equation (4).

The procedure employed above cannot be applied when $p = 2$ since $x_1^2 + cx_1$ consists of $s/2$ elements of $GF(2^m)$, each occurring twice. Thus there exists no element u such that $x_1^2 + cx_1 = u$ must have only one solution.

We deduce from Lemma 2 that when $p = 2$, then $k = 1$. In order that the levels of $X_1^2 - X_2$ which occur with the 0 levels of $X_1 + a_i X_2 + b_i (a_i = 1, 2, 3, \ldots, s-1)$ complement the levels of $X_1^2 + X_2$ which occur with the 0 levels of $X_1 + a_i X_2$ then the levels given by $x_1^2 - (1/a_i)x_1 - b_i/a_i$ must complement the levels given by $x_1^2 - (1/a_i)x_1$ when x_1 ranges over $GF(2^m)$. It is easily verified that b_i can be any one of the 2^{m-1} elements of $GF(2^m)$ which are not given by $x_1^2 - (1/a_i)x_1$.

In order that the levels of $X_1^2 + k_i X_1 + X_2 + c_i$ which occur with the 0 levels of X_2 complement the levels of $X_1^2 + d_i X_1 + X_2$ which occur with the 0 levels of X_2, then the values given by $x_1^2 + k_i x_1 + c_i$ must complement the values given by $x_1^2 + d_i x_1$. It can be shown that $k_i = d_i$ and c_i can be any one of the 2^{m-1} elements of $GF(2^m)$ which are not given by the values of $x_1^2 + d_i x_1$.

By finding the values of $X_1^2 + k_i X_1 + X_2 + c_i$ which occur with the 0 levels of $X_1 + a_i X_2 + b_i$ and which complement the values of $X_1^2 + d_i X_1 + X_2$ that occur with the 0 levels of $X_1 + a_i X_2$, a set of b_i and c_i which satisfy all the requirements to have the second half of the plan complement the first half of the plan can be determined.

When $n > 2$ the same procedures will yield the desired plans if Lemma 5a is utilized in place of Lemma 5. Thus the theorem is proved.

The orthogonal main-effect plans for $[2(s^n-1)/(s-1) - 1]$ factors each at $s = p^m$ levels with $2s^n$ treatment combinations which are included in the catalogue are the following: 3^7 in 18, 3^{25} in 54, 4^9 in 32, 5^{11} in 50. Bose and Bush (1952) have constructed the plans for 3^7 in 18 and 4^9 in 32 by other procedures and have shown that $[2(s^n-1)/(s-1)-1]$ is the maximum number of factors, each at s levels, that can be accommodated in an orthogonal main-effect plan with $2s^n$ treatment combinations.

C. Condition of Proportional Frequencies

In the complete factorial experiment the levels of a factor occur equally frequently with each of the levels of any other factor. This condition is sufficient to allow uncorrelated estimates of all effects and interactions. This condition is also sufficient to allow uncorrelated estimates of the main effects in a main-effect plan. However for main-effect plans the condition of equal frequencies is not a necessary one. We will show that a necessary and sufficient condition that the estimates of the main effects of any two factors in a main-effect plan be uncorrelated is that the levels of one factor occur with each of the levels of the other factor with proportional frequencies. The condition of proportional frequencies, will be deduced for a main-effect plan on two factors, A and B, occurring at r and s levels, respectively. This was stated first, it is believed, by Plackett (1946) but his proof was found to be obscure. Therefore a related proof is presented below.

If the plan is orthogonal then the estimate of any component of factor A is orthogonal with the estimate of any component of factor B. Let the components of factor A be represented by $(r-1)$ orthogonal contrasts, and the components of factor B by $(s-1)$ orthogonal contrasts. Denote by A_u and B_v the u-th orthogonal contrast among the r levels of factor A and the v-th orthogonal contrast among the s levels of factor B, respectively. Denote by $a_{1u}, a_{2u}, \ldots, a_{ru}$ the coefficients of A_u, and by $b_{1v}, b_{2v}, \ldots, b_{sv}$ the coefficients of B_v. The model which exhibits these orthogonal contrasts is

$$y_{ij} = \mu + \sum_{u=1}^{r-1} a_{iu} A_u + \sum_{v=1}^{s-1} b_{jv} B_v + e_{ij};\ i = 0, 1, 2, \ldots, (r-1);$$

$j = 0, 1, 2, \ldots, (s-1)$, where y_{ij} is the observed yield of the treatment combination for which factor A occurs at the i level and factor B occurs at the j level, μ is the overall mean and e_{ij} is the experimental error associated with the observed yield y_{ij}.

Let n = the number of trials in the plan,

$n_{i.}$ = the number of times the i level of factor A occurs in the plan,

$n_{.j}$ = the number of times the j level of factor B occurs in the plan,

n_{ij} = the number of times the i level of factor A occurs with the j level of factor B.

Hence $\sum_j n_{ij} = n_{i.}$, $\sum_i n_{ij} = n_{.j}$ and $\sum_{i,j} n_{ij} = n$.

Theorem 2: A necessary and sufficient condition that the estimates of the components of two factors A and B, in a main-effect plan, be orthogonal to each other and also to the mean μ is that $n_{ij} = n_{i.}\, n_{.j}/n$.

Proof: In order that the estimates of the components of factors A and B be orthogonal to each other and also to the mean, the design matrix X must be such that X'X is a diagonal matrix. With the model

$$y_{ij} = \mu + \sum_{u=1}^{r-1} a_{iu} A_u + \sum_{v=1}^{s-1} b_{jv} B_v + e_{ij};\ i = 0, 1, 2, \ldots, (r-1);$$

$j = 0, 1, 2, \ldots, (s-1)$, the following equations must hold in order that the design matrix be of the required form:

$$\sum_i a_{iu} n_{i.} = 0 ; u = 1, 2, \ldots, (r-1) \tag{5}$$

$$\sum_j b_{jv} n_{.j} = 0 ; v = 1, 2, \ldots, (s-1) \tag{6}$$

$$\sum_i a_{iu} a_{iu'} n_{i.} = 0 ; u \neq u' \tag{7}$$

$$\sum_j b_{jv} b_{jv'} n_{.j} = 0 ; v \neq v' \tag{8}$$

and $$\sum_{i,j} a_{iu} b_{jv} n_{ij} = 0 ; u = 1, 2, \ldots, (r-1) ; v = 1, 2, \ldots, (s-1). \tag{9}$$

Equations (5), (6) and (9) can be expressed in matrix notation by equations (10), (11) and (12):

$$A'N_{r.} = \theta_{r-1,1} \tag{10}$$

where A' is an $(r-1) \times r$ matrix of coefficients of A_u, $N'_{r.} = (n_{0.}\ n_{1.}\ \cdots\ n_{(r-1).})$, and θ_{mn} is an $m \times n$ matrix of zeros;

$$B'N_{.s} = \theta_{s-1,1} \tag{11}$$

where B' is an $(s-1) \times s$ matrix of coefficients of B_v, and $N'_{.s} = (n_{.0}\ n_{.1}\ \cdots\ n_{.(s-1)})$; and

$$A'NB = \theta_{r-1,s-1} \tag{12}$$

where $N = (n_{ij})$.

Equations (7) and (8) are automatically satisfied since the a_{iu} and the b_{jv} are coefficients of the orthogonal contrasts. Thus we need only show that a necessary and sufficient condition that $A'NB = \theta_{r-1,s-1}$, given that $A'N_{r.} = \theta_{r-1,1}$, and $B'N_{.s} = \theta_{s-1,1}$, is that $n_{ij} = n_{i.} r_{.j}/n$, which expressed in matrix notation is $N = N_{r.} N'_{.s}/n$.

To show that this condition is sufficient, assume that $N = N_{r.} N'_{.s}/n$. Then $A'NB = A'N_{r.} N'_{.s}B/u = \theta_{r-1,s-1}$, since $A'N_{r.} = \theta_{r-1,1}$ and $N'_{s.} B = \theta_{1,s-1}$.

To show that this condition is also necessary, assume that $A'NB = \theta_{r-1,s-1}$. Since $n_{i.} = \sum_j n_{ij}$ and $n_{.j} = \sum_i n_{ij}$, then $N_{r.} = NE_{s1}$ and $N'_{.s} = E_{1r} N$, where E_{mn} is an $m \times n$ matrix whose elements are all unity. Let $P = \left[E_{r1} \vdots A\right]$. Since the columns of A are the coefficients of $(r-1)$ orthogonal contrasts, P must be non-singular. Let $Q = \left[E_{s1} \vdots B\right]$. Since the columns of B are the coefficients of $(s-1)$ orthogonal contrasts, Q must be non-singular.

$$\text{Now} \quad P'NQ = \begin{bmatrix} E_{1r} \\ \cdots \\ A' \end{bmatrix} N \left[E_{s1} \vdots B\right]$$

$$= \begin{bmatrix} n & N'_{.s}B \\ A'N_{r.} & A'NB \end{bmatrix} = \begin{bmatrix} n & \theta_{1,s-1} \\ \theta_{r-1,1} & \theta_{r-1,s-1} \end{bmatrix}$$

Thus $P'NQ$ is of rank one. Since P and Q are both non-singular matrices, N must have a rank of one. Hence each row of N is a multiple of the first row and each column is a multiple of the first column. Therefore $n_{ij}/n_{i.} = n_{.j}/n$ or $n_{ij} = n_{i.} n_{.j}/n$ which can be expressed in matrix notation as $N = N_{r.} N'_{.s}/n$.

The theorem can easily be generalized to prove that a necessary and sufficient condition that the estimates of the components of k factors in a main-effect plan be pairwise orthogonal and also orthogonal to the mean

μ is that the levels of each factor occur with the levels of any other factor with proportional frequencies. This generalization can be made by showing that for any pair of factors the proportional frequency condition is both necessary and sufficient to yield orthogonal estimates.

D. Plans for Asymmetrical Factorial Experiments

If the levels of each factor are arranged so that they occur with the levels of any other factor with proportional frequencies, it is possible to derive new classes of orthogonal main-effect plans for asymmetrical factorial experiments. One such class permits the estimation of all main effects without correlation for an experiment involving t_1 factors at s_1 levels, t_2 factors at s_2 levels, up to t_k factors at s_k levels, with s_1^n trials, where s_1 is a prime or the power of a prime, $s_1 > s_2 > \ldots > s_k$ and

$$\sum_{i=1}^{k} t_i \leq (s_1^n - 1)/(s_1 - 1).$$

A method of constructing an orthogonal main-effect plan for the $s_1^{t_1} \times s_2^{t_2} \times \ldots \times s_k^{t_k}$ experiment in s_1^n trials involves collapsing factors occurring at s_1 levels to factors occurring at s_i levels $(i = 2, 3, 4, \ldots, k)$ by utilizing a many-one correspondence of the set of s_1 levels to the set of s_i levels. First construct an orthogonal main-effect plan for the symmetrical factorial experiment involving $(s_1^n - 1)/(s_1 - 1)$ factors, each at s_1 levels, with s_1^n trials, where s_1 is a prime or the power of a prime. Collapse the levels of t_2 of these factors to s_2 levels, where $s_2 < s_1$, by making a many-one correspondence of the set of s_1 levels to the set of s_2 levels. Similarly collapse the levels of t_3 of the original factors to s_3 levels, where $s_3 < s_2 < s_1$, and so on.

If for some i, $s_1 = s_i^m$, then a factor with s_1 levels can be collapsed into $(s_1-1)/(s_i-1)$ factors each having s_i levels. Since there exists an orthogonal main-effect plan for $(s_i^m-1)/(s_i-1)$ factors, each at s_i levels, with s_i^m treatment combinations, we can replace each of the s_1 levels by one of the $s_1 = s_i^m$ treatment combinations. To illustrate this point consider a factor at $s_1 = 4$ levels. There exists an orthogonal main-effect plan for three factors, each having two levels, in four treatment combinations, namely: 0 0 0, 0 1 1, 1 0 1 and 1 1 0. If we make the following correspondence:

Four-level factor		Two-level factors
0	$\longrightarrow$	0 0 0
1	$\longrightarrow$	0 1 1
2	$\longrightarrow$	1 0 1
3	$\longrightarrow$	1 1 0

the four-level factor is collapsed to three two-level factors.

If the (s_i-1) degrees of freedom for each of the t_i factors at s_i levels are represented by (s_i-1) orthogonal contrasts among the s_i levels, the estimates of these contrasts for any factor will be uncorrelated with the estimates of the contrasts for any other factor because the correspondence scheme automatically guarantees proportional frequencies of the levels of each factor.

An orthogonal main-effect plan for the $2^2 \times 3^2$ factorial experiment with nine trials is now constructed to illustrate the technique of collapsing levels. First construct an orthogonal main-effect plan fo four factors, each having three levels with nine treatment combinations.

0 0 0 0
0 1 1 2
0 2 2 1
1 0 1 1
1 1 2 0
1 2 0 2
2 0 2 2
2 1 0 1
2 2 1 0

Collapse each of the first two factors to two-level factors using the following correspondence scheme:

Three-level factor		Two-level factor
0	$\longrightarrow$	0
1	$\longrightarrow$	1
2	$\longrightarrow$	0

The resulting treatment combinations constitute an orthogonal main-effect plan for the $2^2 \times 3^2$ experiment and are displayed below.

0 0	0 0	1 0	0 2
0 1	1 2	0 0	2 2
0 0	2 1	0 1	0 1
1 0	1 1	0 0	1 0
1 1	2 0		

Doubling the number of trials and doubling the number of levels of one factor leads also to some new orthogonal main-effect plans. To illustrate the construction procedure consider the 3^4 plan in 9 observations and repeat it, replacing the levels 0, 1 and 2 in one of the factors by the levels 3, 4 and 5. This gives a 6×3^3 plan in 18 trials. The collapsing procedure will then give the $5 \times 3^{n_1} \times 2^{n_2}$ plan, $\sum_{i=1}^{2} n_i = 3$, in 18 trials.

The class of orthogonal main-effect plans for the $s_1^{t_1} \times s_2^{t_2} \times \ldots \times s_k^{t_k}$ experiment with s_1^n trials where $s_1 > s_2 > \ldots > s_k$ restricts the number of trials to be equal to s_1^n where s_1 is the largest number of levels. Thus, for example, one would require sixteen trials in order to construct an orthogonal main-effect plan for the 4×2^4 experiment using the procedures suggested above. A second class of orthogonal main-effect plans can be derived for the $s_1^{t_1} \times s_2^{t_2} \times \ldots \times s_k^{t_k}$ experiment in s_1^n trials, where s_1 is a prime or the power of a prime, $s_1 \leq s_2 < \ldots < s_k$, and $\sum_{i=1}^{k} \lambda_i t_i < (s_1^n - 1)/(s_1 - 1)$ where $1 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_k$, the λ_i being integers. An orthogonal main-effect plan of this class exists for the 4×2^4 experiment with only eight trials.

Theorem 3: Consider an orthogonal main-effect plan for $(s^n - 1)/(s - 1)$ factors, each at s levels, with s^n trials, where s is a prime or the power of a prime number. Then a factor at t levels, where $s < t \leq s^2$, can be introduced as a replacement for a suitably chosen set of $(s + 1)$ factors in such a way as to preserve orthogonality of main-effect estimates.

Proof: Let $t = s^2$. There exists an orthogonal main-effect plan for $(s^2 - 1)/(s - 1)$ factors each at s levels in $t = s^2$ trials. Hence a factor having $t = s^2$ levels can replace $(s^2 - 1)/(s - 1) = (s + 1)$ factors each having s levels. If $t < s^2$ then a factor having s^2 levels can replace $(s+1)$ factors each having s levels and then collapsed to a t-level factor by a many-one correspondence scheme.

Corollary 1: The maximum number of t-level factors $(s < t \leq s^2)$ which can be introduced into an orthogonal main-effect plan for $(s^n-1)/(s-1)$ factors, each at s levels, with s^n trials is (i) $(s^n-1)/(s^2-1)$ if n is even and (ii) the largest integer less than or equal to $[(s^n-1)/(s^2-1) - 1]$ if n is odd.

Corollary 2: A factor at t levels, where $s^{m-1} < t \leq s^m$ can be introduced as a replacement for a suitably chosen set of $(s^m-1)/(s-1)$ factors each having s levels in such a way as to preserve the orthogonality of main-effect estimates.

This replacement procedure will be illustrated by constructing an orthogonal main-effect plan for the 4×2^4 experiment in eight trials. First construct an orthogonal main-effect plan for the 2^7 experiment. The seven two-level factors can be represented by X_1, X_2, $X_1 + X_2$, X_3, $X_1 + X_3$, $X_2 + X_3$ and $X_1 + X_2 + X_3$. The treatment combinations for this plan are the following:

0 0 0 0 0 0 0

0 0 0 1 1 1 1

0 1 1 0 0 1 1

0 1 1 1 1 0 0

1 0 1 0 1 0 1

1 0 1 1 0 1 0

1 1 0 0 1 1 0

1 1 0 1 0 0 1

It is known that there exists an orthogonal main-effect plan for the 2^3 experiment in four trials. The treatment combinations for this plan are (0 0 0), (0 1 1), (1 0 1) and (1 1 0). Thus by choosing three factors of the 2^7 plan whose X representations are such that the generalized interaction of any two of the three factors is the third factor, these two-level factors can be replaced by a four-level factor according to the following correspondence scheme

Two-level factors		Four-level factor
0 0 0	$\longrightarrow$	0
0 1 1	$\longrightarrow$	1
1 0 1	$\longrightarrow$	2
1 1 0	$\longrightarrow$	3

Since the X representations of the first three factors of the above plan are X_1, X_2 and $X_1 + X_2$, these three factors can be replaced by a four-level factor and the orthogonal main-effect plan for the 4×2^4 experiment in eight trials is given by the following treatment combinations:

0	0 0 0 0
0	1 1 1 1
1	0 0 1 1
1	1 1 0 0
2	0 1 0 1
2	1 0 1 0

3 0 1 1 0

3 1 0 0 1

By collapsing the four-level factor to a three-level factor, an orthogonal main-effect plan for the 3×2^4 experiment is obtained.

It can be easily verified that a suitably chosen set of $(s^2-1)/(s-1)$ factors, each having s^3 levels, occurring in an orthogonal main-effect plan with s^6 trials can be replaced by $(s^3-1)/(s-1)$ factors, each having s^2 levels. This proposition can be illustrated by replacing three eight-level factors by seven four-level factors in an orthogonal main-effect plan with sixty-four trials.

Consider the orthogonal main-effect plan for the 2^{63} experiment in sixty-four trials. Let each factor be represented by an effect or interaction of the 2^6 factorial experiment, namely X_1, X_2, X_3, X_4, X_5, X_6 or any one of their generalized interactions. From Corollary 2 of Theorem 3 it is clear that each eight-level factor introduced into the plan replaces seven two-level factors. Let us denote three eight-level factors by A, B and C. Table 12 gives the X representations for the two-level factors which are replaced by the three eight-level factors.

It will be noted in Table 12 that the X representations of the two-level factors which are replaced by the eight-level factor C are the generalized interactions of the X representations of the two-level factors which are replaced by factors A and B. Thus each row of the table represents three two-level factors which can be replaced by a four-level factor. Hence it is clear that three eight-level factors can be replaced by seven four-level factors.

TABLE 12

TWO-LEVEL FACTORS REPLACED BY EIGHT-LEVEL FACTORS

A	B	C
X_1	X_2	X_1+X_2
X_2+X_4	X_1+X_5	$X_1+X_2+X_4+X_5$
$X_1+X_2+X_4$	$X_1+X_2+X_5$	X_4+X_5
X_3+X_5	$X_1+X_2+X_3+X_5+X_6$	$X_1+X_2+X_6$
$X_1+X_3+X_5$	$X_1+X_3+X_5+X_6$	X_6
$X_2+X_3+X_4+X_5$	$X_2+X_3+X_6$	$X_4+X_5+X_6$
$X_1+X_2+X_3+X_4+X_5$	X_3+X_6	$X_1+X_2+X_4+X_5+X_6$

It is evident that the use of factors for which the levels occur with proportional frequencies also yields orthogonal main-effect plans for symmetrical factorial experiments. For example, an orthogonal main-effect plan for the 3^5 experiment can be constructed with sixteen trials by collapsing all the four-level factors in the plan for the 4^5 experiment to three-level factors.

The use of factors whose levels occur with proportional frequencies also permits the construction of orthogonal main-effect plans for factors for which the number of levels is not equal to a prime or the power of a prime. One such plan with forty-nine trials permits uncorrelated main-effect estimates for the 6^8 experiment. This plan can be constructed by collapsing the seven-level factors to six-level factors in the plan for the 7^8 experiment.

E. Efficiencies of Main-Effect Estimates

Although any many-one correspondence of the set of s_1 levels to the set of s_i levels will yield proportional frequencies of the levels, there arises the problem of which correspondence is "best" in some sense. The problem may be solved by determining the efficiencies of the main-effect estimates obtained using proportional frequencies relative to the estimates which would result from using equal frequencies of the levels of each factor.

As an illustration we will calculate the relative efficiency of a three-level factor in a main-effect plan with twenty-five trials.

Assume the correspondence scheme used to collapse a five-level factor to three levels is as follows:

Five-level factor		Three-level factor
0	$\longrightarrow$	0
1	$\longrightarrow$	1
2	$\longrightarrow$	2
3	$\longrightarrow$	2
4	$\longrightarrow$	0

The levels 0, 1, and 2 occur in the ratio's 2 : 1 : 2. Thus for this factor the 0 level occurs in ten treatment combinations, the 1 level occurs in five treatment combinations and the 2 level occurs in ten treatment combinations.

The variance of the linear effect estimate of this factor is equal to $\sigma^2/20$ and hence the information on a unit basis is equal to $20/25\sigma^2 = 4/5\sigma^2$. The variance of the linear effect estimate of a three-level factor in 3^n trials is equal to $\sigma^2/2.3^{n-1}$ and the information on a unit basis is $2.3^{n-1}/3^n\sigma^2 = 2/3\sigma^2$. Hence the relative efficiency of the linear effect estimate is equal to $4/5 \times 3/2 = 6/5$.

The variance of the quadratic effect estimate for the three-level factor in twenty-five trials is equal to $\sigma^2/4$ and the information is then equal to $4/25\sigma^2$. The variance of the quadratic effect estimate with 3^n trials is equal to $\sigma^2/2.3^{n-2}$ and hence the information on a unit basis is equal to $2/9\sigma^2$. The relative efficiency of the quadratic effect estimate is therefore equal to $4/25 \times 9/2 = 18/25$.

The relative efficiencies of the estimated effects are presented for various proportional frequencies in Table 13 . One would chose the proportional frequencies which give the greatest efficiency of estimates. Thus for example, if an experiment in twenty-five trials involved two-level factors the two levels should occur in the ratio 2 : 3 rather than in the ratio 1 : 4 because the efficiency of the 2 : 3 ratio is 24/25 whereas the efficiency of the 1 : 4 ratio is only 16/25.

TABLE 13

RELATIVE EFFICIENCIES OF MAIN-EFFECT ESTIMATES

Level	0 1	Efficiency
	Proportional frequency	
	1 : 2	8/9
	2 : 3	24/25
	1 : 4	16/25
	3 : 4	48/49
	2 : 5	40/49
	1 : 6	24/49
Level	0 1 2	
Contrast	Proportional frequency	
Linear	1 : 2 : 1	3/4
Quadratic	1 : 2 : 1	9/8
Linear	2 : 1 : 2	6/5
Quadratic	2 : 1 : 2	18/25
Linear	1 : 3 : 1	3/5
Quadratic	1 : 3 : 1	27/25
Linear	2 : 3 : 2	6/7
Quadratic	2 : 3 : 2	54/49
Linear	3 : 1 : 3	9/7
Quadratic	3 : 1 : 3	27/49
Linear	1 : 5 : 1	3/7
Quadratic	1 : 5 : 1	45/49

F. Blocking in Main-Effect Plans

Even though the orthogonal main-effect plans are highly fractionated these plans may still require more trials than can be carried out under uniform conditions. Thus it would be desirable to divide the experimental data into smaller blocks in such a manner that the main effects may still be estimated without correlation. In this section we will illustrate the use of some of the factors in an orthogonal main-effect plans as blocking factors.

Consider the orthogonal main-effect plan for the 3^4 experiment with nine trials. The treatment combinations for this plan are

0 0 0 0

0 1 1 2

0 2 2 1

1 0 1 1

1 1 2 0

1 2 0 2

2 0 2 2

2 1 0 1

2 2 1 0

If there are only three factors under investigation the fourth factor of the above plan can be used as a blocking factor to yield the following blocks:

Block 1	Block 2	Block 3
0 0 0	0 2 2	0 1 1
1 1 2	1 0 1	1 2 0
2 2 1	2 1 0	2 0 2

The estimate of the main effects of the three factors are clear of the block effects since each level of the three factors occurs once in each block. The linear effect of the first factor is given by

$$\frac{1}{6}(2\,2\,1 + 2\,1\,0 + 2\,0\,2 - 0\,0\,0 - 0\,2\,2 - 0\,1\,1).$$

It is evident that each block effect enters this estimate once positively and once negatively and hence the estimate is clear of block effects.

If the four factors in the orthogonal main-effect plan are represented by X_1, X_2, $X_1 + X_2$ and $X_1 + 2X_2$ the use of the fourth factor as a blocking factor is equivalent to confounding the factor represented by $X_1 + 2X_2$ with blocks.

In general, if two factors are used as blocking factors then so are the factors represented by the generalized interactions of their X representations. For example, if the seven factors in the plan for the 2^7 experiment with eight trials are represented by X_1, X_2, $X_1 + X_2$, X_3, $X_1 + X_3$, $X_2 + X_3$ and $X_1 + X_2 + X_3$ an orthogonal main-effect plan for the 2^4 experiment in 4 blocks of 2 treatment combinations can be obtained by using the factors represented by X_1, X_2, $X_1 + X_2$ as blocking factors. This is equivalent to confounding X_1, X_2 and $X_1 + X_2$ with blocks.

Now consider the orthogonal main-effect plan for the $4 \times 3^2 \times 2^6$ experiment with sixteen trials. The plan is comprised of the following treatment combinations:

0 0 0	0 0 0 0 0 0
0 1 1	1 0 1 1 1 0
0 2 2	1 1 0 0 1 1
0 1 1	0 1 1 1 0 1
1 0 1	0 1 1 0 1 1
1 1 0	1 1 0 1 0 1
1 2 1	1 0 1 0 0 0
1 1 2	0 0 0 1 1 0
2 0 2	1 0 1 1 0 1
2 1 1	0 0 0 0 1 1
2 2 0	0 1 1 1 1 0
2 1 1	1 1 0 0 0 0
3 0 1	1 1 0 1 1 0
3 1 2	0 1 1 0 0 0
3 2 1	0 0 0 1 0 1
3 1 0	1 0 1 0 1 1

The following orthogonal main-effect plans which utilize one or more of the factors as blocking factors may be constructed from the given plan:

(i) $4 \times 3^2 \times 2^5$ in 2 blocks of 8 treatment combinations:

Using the last two-level factor as a blocking factor the two blocks consist of the treatment combinations presented below:

Block 1		Block 2	
0 0 0	0 0 0 0 0	0 2 2	1 1 0 0 1
0 1 1	1 0 1 1 1	0 1 1	0 1 1 1 0
1 2 1	1 0 1 0 0	1 0 1	0 1 1 0 1
1 1 2	0 0 0 1 1	1 1 0	1 1 0 1 0
2 2 0	0 1 1 1 1	2 0 2	1 0 1 1 0
2 1 1	1 1 0 0 0	2 1 1	0 0 0 0 1
3 0 1	1 1 0 1 1	3 2 1	0 0 0 1 0
3 1 2	0 1 1 0 0	3 1 0	1 0 1 0 1

(ii) $4 \times 3^2 \times 2^3$ in 4 blocks of 4 treatment combinations:

Consider the last three two-level factors only. The levels for these three factors occur in the four sets 0 0 0, 0 1 1, 1 0 1 and 1 1 0 each occurring four times in the sixteen trials. Th treatment combinations of the first six factors can be blocked according to the particular set to which the levels of the last three factors belong. Hence the four blocks are:

Block 1	Block 2	Block 3	Block 4
0 0 0 0 0 0	0 2 2 1 1 0	0 1 1 0 1 1	0 1 1 1 0 1
1 2 1 1 0 1	1 0 1 0 1 1	1 1 0 1 1 0	1 1 2 0 0 0
2 1 1 1 1 0	2 1 1 0 0 0	2 0 2 1 0 1	2 2 0 0 1 1
3 1 2 0 1 1	3 1 0 1 0 1	3 2 1 0 0 0	3 0 1 1 1 0

(iii) $3^2 \times 2^6$ in 4 blocks of 4 treatment combinations:

Utilizing the four-level factor as the blocking factor the four blocks are:

Block 1	Block 2	Block 3	Block 4
0 0 0 0 0 0 0 0	0 1 0 1 1 0 1 1	0 2 1 0 1 1 0 1	0 1 1 1 0 1 1 0
1 1 1 0 1 1 1 0	1 0 1 1 0 1 0 1	1 1 0 0 0 0 1 1	1 2 0 1 1 0 0 0
2 2 1 1 0 0 1 1	2 1 1 0 1 0 0 0	2 0 0 1 1 1 1 0	2 1 0 0 0 1 0 1
1 1 0 1 1 1 0 1	1 2 0 0 0 1 1 0	1 1 1 1 0 0 0 0	1 0 1 0 1 0 1 1

(iv) $4 \times 3 \times 2^6$ in 4 blocks of 4 treatment combinations:

This plan can be constructed by considering the first three-level factor to be a four level factor and using that factor as a blocking factor. If every second 1 in the first three-level factor of the main-effect plan for the $4 \times 3^2 \times 2^6$ experiment is replaced by a 3 the sixteen treatment combinations then comprise a main-effect plan for the $4^2 \times 3 \times 2^6$ experiment.

Block 1	Block 2	Block 3	Block 4
0 0 0 0 0 0 0 0	0 1 1 0 1 1 1 0	0 2 1 1 0 0 1 1	0 1 0 1 1 1 0 1
1 1 0 1 1 0 1 1	1 0 1 1 0 1 0 1	1 1 1 0 1 0 0 0	1 2 0 0 0 1 1 0
2 2 1 0 1 1 0 1	2 1 0 0 0 0 1 1	2 0 0 1 1 1 1 0	2 1 1 1 0 0 0 0
3 1 1 1 0 1 1 0	3 2 0 1 1 0 0 0	3 1 0 0 0 1 0 1	3 0 1 0 1 0 1 1

It is clear that in each of the above plans, the main-effect estimates and the block effect estimates are uncorrelated.

G. Randomization Procedure

An important aspect of most experimental situations is the fact that each experimental unit can be subjected to only one of the treatments of interest. Because of this fact the variability due to heterogeneity of experimental units will contribute to experimental uncertainty. To obtain some control of this variability the device of randomization is used in the statistical design of experiments. This technique implies, essentially, that random methods of selection and assignment are employed in carrying out the experiment.

The procedure recommended for assigning treatments at random to the experimental units of an orthogonal main-effect plan is as follows:

(i) Choose the appropriate plan.

(ii) Randomly assign the factors to the columns of the chosen plan.

(iii) Randomly assign the levels of each factor to the numbers 0, 1, 2, ..., representing the levels of a factor.

(iv) Randomly assign the treatments to the experimental units.

To illustrate this procedure we shall describe the randomization procedure to be followed with an experiment involving three factors A, B and C, each having three levels and one factor, D, at two levels. The appropriate orthogonal main-effect plan for this experiment is given by the following nine treatment combinations.

0	0	0	0
0	1	1	0
0	2	2	1
1	0	1	1
1	1	2	0
1	2	0	0
2	0	2	0
2	1	0	1
2	2	1	0

Assign the factors A, B and C at random to the first three columns of the above plan and assign factor D to the fourth column. Then, for each of the factors A, B and C randomly assign the three levels to 0, 1 and 2. Similarly for factor D assign the two levels to 0 and 1 at random. Then these treatments are assigned to nine experimental units at random, for example, by testing the combinations in random order.

H. Analysis of Main-Effect Experiments

An important feature of the full factorial arrangement is that the main effects and all interactions can be estimated without correlation Since the main-effect plans developed in this report allow uncorrelated estimates of all main effects the analyses of these experiments are similar to the analysis of a complete factorial experiment. Estimation is based on the general procedure described in Chapter II, and a quick review of aspects relevant to main effect plans will now be given.

The multiple regression model for an orthogonal main-effect experiment can be written in matrix notation as $y = X\beta + e$ where β is the vector of effects and interactions. The estimates of the effects and interactions are given by $\hat{\beta} = (X'X)^{-1} X'y$, where $(X'X)^{-1}$ is the variance-covariance matrix. The property of uncorrelated estimates is reflected in the fact that the variance-covariance matrix is a diagonal matrix.

To illustrate the estimation procedure we consider the plan for two two-level factors, A and B, and two three-level factors, C and D, the levels being equally spaced, in nine trials, when all interactions are assumed to be absent. The plan is given by the following set of treatment combinations:

0	0	0	0
0	1	1	2
0	0	2	1
1	0	1	1
1	1	2	0
1	0	0	2
0	0	2	2
0	1	0	1
0	0	1	0

The responses y_{ijkm} may be expressed in terms of the main effects as

$$y_{ijkm} = \mu + a_i A + b_j B + c_{k_L} C_L + c_{k_Q} C_Q + d_{m_L} D_L + d_{m_Q} D_Q + e_{ijkm}$$

where A, B, C_L, C_Q, D_L and D_Q are the effects of the respective factors and a_i, b_j, c_{k_L}, c_{k_Q}, d_{m_L}, d_{m_Q} are the coefficients of the orthogonal contrasts defining the corresponding effects. The factors are assumed to be quantitative factors and the levels of the factors are equally spaced.*

If any of the factors are qualitative, they can still be treated as quantitative factors, except that what are contrasts of specific meaning in the quantitative case, such as linear and quadratic effects, are merely contrasts among the levels of the qualitative factors. For example, if we use the numbers 0, 1 and 2 to denote the levels of a factor, F, at three levels and get what look superficially to be linear and quadratic effects, they are in fact

$$L = F_2 - F_0$$

$$Q = F_2 - 2F_1 - F_0 = (F_2 - F_1) - (F_1 - F_0)$$

where L and Q denote the linear and quadratic effects and F_i denotes the treatment combinations which contain factor F at the i level. From such calculated effects one can determine any contrasts which seem relevant. For instance

$$F_2 - F_1 = (L + Q)/2$$

and

$$F_1 - F_0 = (L - Q)/2 .$$

*If the levels of quantitative factors are not at equally spaced intervals the effects can still be written in terms of orthogonal contrasts. A procedure for obtaining orthogonal polynomials for unequally spaced levels is given in section C of Chapter V.

The matrix of known coefficients* is given by

$$
\begin{array}{cccccccc}
 & \mu & \frac{1}{3}A & \frac{1}{3}B & C_L & \frac{1}{3}C_Q & D_L & \frac{1}{3}D_Q \\
X = & \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}\right. & \begin{array}{c} -1 \\ -1 \\ -1 \\ 2 \\ 2 \\ 2 \\ -1 \\ -1 \\ -1 \end{array} & \begin{array}{c} -1 \\ 2 \\ -1 \\ -1 \\ 2 \\ -1 \\ -1 \\ 2 \\ -1 \end{array} & \begin{array}{c} -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ -1 \\ 1 \\ -1 \\ 0 \end{array} & \begin{array}{c} 1 \\ -2 \\ 1 \\ -2 \\ 1 \\ 1 \\ 1 \\ 1 \\ -2 \end{array} & \begin{array}{c} -1 \\ 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 1 \\ 0 \\ -1 \end{array} & \left.\begin{array}{c} 1 \\ 1 \\ -2 \\ -2 \\ 1 \\ 1 \\ 1 \\ -2 \\ 1 \end{array}\right]
\end{array}
$$

Hence the information matrix is

$$
X'X = \begin{bmatrix}
9 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 18 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 18 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 18 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 18
\end{bmatrix}
$$

---

*The coefficients of the parameters are obtained by a convention which is discussed in section C of Chapter V.

Since $X'X$ is a diagonal matrix the plan is orthogonal and the variance-covariance matrix is given by

$$(X'X)^{-1} = \frac{1}{18}\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The estimates of effects and interactions are

$$\begin{bmatrix} \hat{\mu} \\ \frac{1}{3}\hat{A} \\ \frac{1}{3}\hat{B} \\ \hat{C}_L \\ \frac{1}{3}\hat{C}_Q \\ \hat{D}_L \\ \frac{1}{3}\hat{D}_Q \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 \\ -3 & 0 & 3 & 0 & 3 & -3 & 3 & -3 & 0 \\ 1 & -2 & 1 & -2 & 1 & 1 & 1 & 1 & -2 \\ -3 & 3 & 0 & 0 & -3 & 3 & 3 & 0 & -3 \\ 1 & 1 & -2 & -2 & 1 & 1 & 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \end{bmatrix}$$

where $y_1, y_2, \ldots, y_9$ are the responses of the nine treatment combinations in the order presented in the plan.

Thus, for example,

$$\tfrac{1}{3}\hat{A} = \tfrac{1}{18}\left[-y_1 - y_2 - y_3 + 2y_4 + 2y_5 + 2y_6 - y_7 - y_8 - y_9\right],$$

and $$C_L = \tfrac{1}{6}\left[-y_1 + y_3 + y_5 - y_6 + y_7 - y_8\right].$$

The variances of the estimates are obtained from the variance-covariance matrix. Thus,

$$\operatorname{var}(\hat{A}) = \operatorname{var}(\hat{B}) = \sigma^2/2$$

$$\operatorname{var}(\hat{C}_L) = \operatorname{var}(\hat{D}_L) = \sigma^2/6$$

$$\operatorname{var}(\hat{C}_Q) = \operatorname{var}(\hat{D}_Q) = \sigma^2/2$$

An unbiased estimate of σ^2 is derived from the sum of squares of deviations about the estimated values, namely

$$\hat{\sigma}^2 = \tfrac{1}{2}(y'y - \hat{\beta}'X'y)$$

The sum of squares in the analysis of variance associated with any contrast is merely the square of the contrast divided by the sum of squares of the coefficients of the contrast. Hence, the sum of squares due to $\frac{1}{3}A$ is

$$\tfrac{1}{18}\left[-y_1 - y_2 - y_3 + 2y_4 + 2y_5 + 2y_6 - y_7 - y_8 - y_9\right]^2$$

The sum of squares due to A is then

$$\tfrac{1}{2}\left[-y_1 - y_2 - y_3 + 2y_4 + 2y_5 + 2y_6 - y_7 - y_8 - y_9\right]^2$$

If the total sum of squares is corrected for the mean, the partitioning for the analysis of variance is given in Table 14.

TABLE 14

PARTITIONING OF ANALYSIS OF VARIANCE

Source	Degrees of Freedom
A	1
B	1
C_L	1
C_Q	1
D_L	1
D_Q	1
Error	2
Total	8

It is clear that the two degrees of freedom available for estimation of error are the result of collapsing two three-level factors to two-level factors. The estimate of error can be partitioned into single degrees of freedom as follows. Consider the levels of factor A and factor B as they were before being collapsed. The levels are then given as

Levels of factor A: 0 0 0 1 1 1 2 2 2

Levels of factor B: 0 1 2 0 1 2 0 1 2

In order to collapse a three-level factor to a two-level factor we make the correspondence

Three-level factor		Two-level factor
0	$\longrightarrow$	0
1	$\longrightarrow$	1
2	$\longrightarrow$	0

If factors A and B were three-level factors then

$$A_L = \frac{1}{6}(-y_1 - y_2 - y_3 + y_7 + y_8 + y_9)$$

and

$$B_L = \frac{1}{6}(-y_1 + y_3 - y_4 + y_6 - y_7 + y_9)$$

Since these two factors have only two levels and the level of factor A is 0 for each response in the estimate A_L and the level of factor B is 0 for each response in the estimate B_L, then these contrasts are estimating pure error. Thus, the two single degrees of freedom estimates of σ^2 are given by

$$\frac{1}{6}(-y_1 - y_2 - y_3 + y_7 + y_8 + y_9)^2$$

and

$$\frac{1}{6}(-y_1 + y_3 - y_4 + y_6 - y_7 + y_9)^2$$

The partitioning of the analysis of variance is presented in Table 15. If several estimates of error are possible one can determine whether they are homogeneous estimates of error (e.g. Bartlett's test) and if they are found to be homogeneous they can be combined to give a pooled estimate of error. Evidence of estimates of error being not poolable is evidence that there are interactions present in the situation, and further experimentation to explore these is needed.

TABLE 15

PARTITIONING OF ANALYSIS OF VARIANCE

Source	Degrees of Freedom
A	1
B	1
C_L	1
C_Q	1
D_L	1
D_Q	1
Error A	1
Error B	1
Total	8

V. CATALOGUE OF ORTHOGONAL MAIN-EFFECT PLANS

A. Construction of Basic Plans

The task of presenting a catalogue which gives every possible orthogonal main-effect plan with 81 trials or fewer is enormous and need not be undertaken. Each of these plans can be easily deduced from one of twenty-six "basic plans", by choosing a suitable set of columns.

Consider the orthogonal main-effect plan for the 3^4 experiment in nine trials. It was demonstrated in Chapter IV that from this plan one can obtain plans for the following experiments: $3^3 \times 2$, $3^2 \times 2^2$, 3×2^3 and 2^4. If a plan which consisted of the plans for both the 3^4 experiment and the 2^4 experiment in nine trials is given, then the plans for any one of the 3^4, $3^3 \times 2$, $3^2 \times 2^2$, 3×2^3 or 2^4 experiments can be obtained by selecting the appropriate number of columns from the plan for the 3^4 and 2^4 experiments, respectively. The plan which consists of the plans for both the 3^4 experiment and the 2^4 experiment in nine trials is called a basic plan.

Similarly a basic plan for the $4^{t_1} \times 3^{t_2} \times 2^{t_3}$ experiment in sixteen trials is a plan consisting of the plans for the 4^5, 3^5 and 2^{15} experiments.

Each column of the basic plan represents a factor. The number of levels of any factor can be determined by counting the number of different symbols 0, 1, 2, etc. which represent the levels. The columns are numbered so that each column may be identified quickly. The numbering of the columns may best be explained by an example. The column numbers on the four-level factors of basic plan 5, which consists of the plans for the 4^5, 3^5 and 2^{15} experiments in sixteen trials, are:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ * & * & * & * & * \end{matrix} .$$

The numbers on the three-level columns for this plan are also

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ * & * & * & * & * \end{matrix} .$$

The numbers on the two-level factors range from 1 to 15 where for tabular convenience these numbers are written in the form

$$\begin{matrix} 0 & 0 & \ldots\ldots & 1 \\ 1 & 2 & & 5 \end{matrix} .$$

The footnote given below the basic plan indicates that the column identified by $\genfrac{}{}{0pt}{}{1}{*}$ replaces the columns identified by $\genfrac{}{}{0pt}{}{0}{1}$ $\genfrac{}{}{0pt}{}{0}{2}$ and $\genfrac{}{}{0pt}{}{0}{3}$, the column identified by $\genfrac{}{}{0pt}{}{2}{*}$ replaces the columns identified by $\genfrac{}{}{0pt}{}{0}{4}$ $\genfrac{}{}{0pt}{}{0}{5}$ and $\genfrac{}{}{0pt}{}{0}{6}$ and so on. Hence, if a four-level factor identified by $\genfrac{}{}{0pt}{}{1}{*}$ is used in an orthogonal main-effect plan then the three-level-factor identified by column $\genfrac{}{}{0pt}{}{1}{*}$ and the three two-level factors identified by columns $\genfrac{}{}{0pt}{}{0}{1}$ $\genfrac{}{}{0pt}{}{0}{2}$ and $\genfrac{}{}{0pt}{}{0}{3}$ cannot be used.

B. Use of the Catalogue

In this section we will illustrate the use of the catalogue with several examples.

(i) $\underline{2^{10}}$:

The index of orthogonal main-effect plans given in section D of this chapter indicates that a plan can be obtained for the 2^{10} experiment in twelve trials from basic plan 4. The basic plan has twelve treatment combinations and eleven factors. Choose any ten columns of this plan and the required plan is obtained.

(ii) $\underline{3 \times 2^3}$:

The plan for the 3×2^3 experiment in eight trials can be determined from basic plan 2. The footnote to this plan indicates that if the three-level factor is chosen then the two-level factors numbered 1, 2 and 3 cannot be chosen. Thus the plan is obtained by choosing the column representing the three-level factor and any three of the four columns 4, 5, 6 and 7.

A plan for the 3×2^3 experiment in nine trials is given by basic plan 3. The plan can be obtained by choosing column 1 from the three-level factors and columns 2, 3 and 4 from the two-level factors. It is clear that a plan for the 3×2^3 experiment can be obtained from basic plan 3 by choosing any one of the four columns for three-level factors and three columns from the two-level factors, no column of the two-level factor having the same column number as the column number of the chosen three-level factor.

(iii) $\underline{4^2 \times 3 \times 2^5}$:

The index indicates that an orthogonal main-effect plan for the $4^2 \times 3 \times 2^5$ experiment can be constructed in sixteen trials from basic plan 5. The plan may consist of the columns numbered $\genfrac{}{}{0pt}{}{1}{*}$ and $\genfrac{}{}{0pt}{}{2}{*}$ from the four-level columns, $\genfrac{}{}{0pt}{}{3}{*}$ from the three-level columns and $\genfrac{}{}{0pt}{}{1}{0}$, $\genfrac{}{}{0pt}{}{1}{1}$, $\genfrac{}{}{0pt}{}{1}{2}$, $\genfrac{}{}{0pt}{}{1}{3}$ and $\genfrac{}{}{0pt}{}{1}{4}$ from the two-level columns. The use of the four-level columns $\genfrac{}{}{0pt}{}{1}{*}$ and $\genfrac{}{}{0pt}{}{2}{*}$ eliminates the use of the three-level columns identified by $\substack{1\\ *}$ and $\substack{2\\ *}$ and also the two-level columns identified by $\substack{0\\ 1}\ \substack{0\\ 2}\ \substack{0\\ 3}\ \substack{0\\ 4}\ \substack{0\\ 5}$ and $\substack{0\\ 6}$. The use of the three-level column $\substack{3\\ *}$ eliminates the use of the two-level columns $\substack{0\\ 7}\ \substack{0\\ 8}$ and $\substack{0\\ 9}$.

(iv) $\underline{8^3 \times 4^7 \times 2^{10}}$:

A plan for the $8^3 \times 4^7 \times 2^{10}$ experiment in sixty-four trials may be deduced from basic plan 25. If the three eight-level factors chosen are the columns identified by $\substack{1\\ \#}\ \substack{2\\ \#}$ and $\substack{3\\ \#}$ then the four-level factors identified by the seven columns numbered

$$\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ * & * & * & * & * & * & * \end{array}$$

and the two-level factors identified by the columns numbered from $\substack{0\\ 1}$ to $\substack{2\\ 1}$ cannot be used. We then can choose the seven four-level factors to be the columns numbered

$$\begin{array}{ccc} 0 & & 1 \\ 8 & \text{to} & 4 \\ * & & * \end{array} .$$

Thus the ten two-level factors must be chosen from the columns numbered $\substack{4\\ 3}$ to $\substack{6\\ 3}$.

C. Tables of Orthogonal Polynomials

The orthogonal contrasts which define effects and interactions can be readily determined from a table of orthogonal polynomials. The advantage of using orthogonal contrasts to define effects and interactions arises from the fact that orthogonal polynomials are so constructed that any term of the polynomial is independent of any other term. This property of independence permits one to compute each regression coefficient independently of the others and also facilitates testing the significance of each coefficient.

Tables of Orthogonal Polynomials for the case of equally spaced levels are readily available, e.g. Fisher and Yates (1938), Anderson and Houseman (1942). It would be an impossible task to compute the orthogonal polynomials for unequally spaced levels. However a simple procedure for computing these orthogonal polynomials is available and will be presented below. If equally spaced levels do not each occur in a plan an equal number of times the published tables of orthogonal polynomials are not appropriate. The orthogonal polynomials for equally spaced levels which do not occur in a plan with equal frequency must be computed by the following method for unequally spaced levels.

For any set of orthogonal polynomials the linear contrast is of the form $\Sigma(\alpha + \beta x)y_x$, where α and β are constants, x is the level at which the factor occurs, y_x is the response to the treatment combination with the factor at the x level and the summation is over every value of x which is represented. The quadratic and cubic contrasts are of the form $\Sigma(\alpha + \beta x + \gamma x^2)y_x$ and $\Sigma(\alpha + \beta x + \gamma x^2 + \delta x^3)y_x$, respectively. The extension to higher order contrasts is obvious. Two contrasts are orthogonal if the coefficients of each contrast sum to zero and the sum of products of the corresponding coefficients of the two contrasts is zero.

We will illustrate the procedure for obtaining orthogonal polynomials for unequally spaced levels with an example.

Consider an independent variable x with levels 0, 1, 2 and 4.

The coefficients of the linear, quadratic and cubic contrasts for this example is displayed in Table 16.

TABLE 16

COEFFICIENTS OF ORTHOGONAL CONTRASTS

Level of x	Linear	Quadratic	Cubic
0	α	α	α
1	$\alpha + \beta$	$\alpha + \beta + \gamma$	$\alpha + \beta + \gamma + \delta$
2	$\alpha + 2\beta$	$\alpha + 2\beta + 4\gamma$	$\alpha + 2\beta + 4\gamma + 8\delta$
4	$\alpha + 4\beta$	$\alpha + 4\beta + 16\gamma$	$\alpha + 4\beta + 16\gamma + 64\delta$

The coefficients of the linear contrast must sum to zero. Thus,

$$4\alpha + 7\beta = 0 .$$

Setting $\beta = 1$ we find that $\alpha = -7/4$. In order that the coefficients of the orthogonal contrasts be integers reduced to lowest terms we multiply these coefficients by 4 to obtain $\beta = 4$ and $\alpha = -7$. Substituting $\alpha = -7$ and $\beta = 4$ in the linear contrast given in Table 16, gives the linear coefficients.

Level of x	Coefficient of linear contrast
0	-7
1	-3
2	1
4	9

The coefficients of the quadratic contrast must sum to zero. Hence,

$$4\alpha + 7\beta + 21\gamma = 0 .$$

The sum of products of the corresponding coefficients of the linear and quadratic contrasts must also equal zero. Thus,

$$35\beta + 145\gamma = 0 .$$

Solving these two equations to obtain integral values for α, β and γ we obtain $\alpha = 14$, $\beta = -29$ and $\gamma = 7$.

If we substitute these values in the quadratic contrast and reduce the resulting coefficients to lowest terms the coefficients of the quadratic contrast is given by

Level of x	Coefficients of Quadratic contrast
0	7
1	-4
2	-8
4	5

Similarly the sum of the coefficients of the cubic contrast and the sum of products of the corresponding coefficients of the linear and cubic contrasts and the quadratic and cubic contrasts must each equal zero. Hence,

$$4\alpha + 7\beta + 21\gamma + 73\delta = 0$$

$$35\beta + 145\gamma + 581\delta = 0$$

$$44\gamma + 252\delta = 0$$

Solving these equations to obtain integral values for α, β, γ and δ we obtain $\alpha = -36$, $\beta = 392$, $\gamma = -315$ and $\delta = 55$. If we substitute these values in the form of the coefficients of the cubic contrast given in Table 16 and reduce the resulting coefficients to lowest terms, the coefficients of the cubic contrast are given by

Level of x	Coefficients of Cubic contrast
0	-3
1	8
2	-6
4	1

The orthogonal polynomials are presented in the following table.

TABLE 17

ORTHOGONAL POLYNOMIALS

Level of x	Linear	Quadratic	Cubic
0	-7	7	-3
1	-3	-4	8
2	1	-8	-6
4	9	5	1

The symbol β represents one unit of the linear effect of a factor when set equal to unity. In order to obtain integral coefficients β was set equal to 4 and hence $\frac{1}{4}\beta$ represents one unit of the linear effect. Consequently the linear contrast with coefficients given in Table 17 represents the estimate of $\frac{1}{4}$ the linear effect of the factor. It is easily verified that the coefficients of the quadratic contrast are given by

$$7 - \frac{29}{2}x + \frac{7}{2}x^2$$

where $x = 0, 1, 2$ and 4, respectively. Thus the symbol $\frac{2}{7}\gamma$ represents one unit of the quadratic effect, and the linear contrast with coefficients given in Table 17 represents the estimate of $\frac{2}{7}$ the quadratic effect of the factor. Similarly it may be demonstrated that the cubic contrast with coefficients given in Table 17 represents the estimate of 12/55 the cubic effect of the factor.

This constant which is multiplying each effect will be denoted by $\frac{1}{\lambda}$ and in the tables of orthogonal polynomials the value of λ and the sum of squares of the coefficients denoted by Σ, will both be given. Thus any contrast defined by the coefficients given in the tables of orthogonal polynomials represents $\frac{1}{\lambda}$ times the appropriate effect of the factor. It was this convention by which the coefficients of the parameters for the example in section H of Chapter IV were calculated.

In the tables of orthogonal polynomials the coefficients of a linear contrast will be denoted by θ_1, the coefficients of a quadratic contrast by θ_2 and so on. The levels of a factor will be denoted by x.

TABLE 18

ORTHOGONAL POLYNOMIALS

x	θ_1	x	θ_1	x	θ_1	x	θ_1
						0	-3
				0	-2	0	-3
		0	-1	0	-2	0	-3
0	-1						
		0	-1	0	-2	0	-3
1	1						
		1	2	1	3	1	4
				.	3	1	4
						1	4
λ	2		3		5		7
Σ	2		6		30		84

TABLE 18 (Continued)

	x	θ_1	θ_2	x	θ_1	θ_2	x	θ_1	θ_2	x	θ_1	θ_2
										0	-1	3
							0	-1	1	0	-1	3
	0	-1	1	0	-1	1	0	-1	1	1	0	-4
	1	0	-2	1	0	-1	1	0	-4	1	0	-4
	2	1	1	1	0	-[illegible]	2	1	1	1	0	-4
				2	1	1	2	1	1	2	1	3
										2	1	3
λ		1	3		1	2		1	5		1	7
Σ		2	6		2	4		4	20		4	84

TABLE 18 (Continued)

x	θ_1	θ_2	θ_3
0	-3	1	-1
1	-1	-1	3
2	1	-1	-3
3	3	1	1
λ	2	1	$\frac{10}{3}$
Σ	20	4	20

x	θ_1	θ_2	θ_3
0	-6	10	-1
0	-6	10	-1
1	-1	-21	6
2	4	-18	-6
3	9	19	2
	5	17	$\frac{13}{2}$
	170	1,326	78

x	θ_1	θ_2	θ_3
0	-11	13	-1
0	-11	13	-1
1	-4	-20	6
2	3	-19	-3
2	3	-19	-3
3	10	16	1
3	10	16	1
	7	17	$\frac{29}{6}$
	476	1,972	58

x	θ_1	θ_2	θ_3
0	-14	24	-1
0	-14	24	-1
1	-5	-21	3
1	-5	-21	3
2	4	-20	-2
2	4	-20	-2
2	4	-20	-2
3	13	27	1
3	13	27	1
	9	23	$\frac{17}{6}$
	828	4,692	34

TABLE 18 (Contin..ed)

x	θ_1	θ_2	θ_3	θ_4
0	-2	2	-1	1
1	-1	-1	2	-4
2	0	-2	0	6
3	1	-1	-2	-4
4	2	2	1	1
λ	1	1	$\frac{5}{6}$	$\frac{35}{12}$
Σ	10	14	10	70

x	θ_1	θ_2	θ_3	θ_4
0	-2	10	-1	1
0	-2	10	-1	1
1	-1	-11	4	-8
2	0	-18	0	12
3	1	-11	-4	-8
4	2	10	1	1
4	2	10	1	1
λ	1	7	$\frac{3}{2}$	$\frac{23}{4}$
Σ	18	966	36	276

x	θ_1	θ_2	θ_3	θ_4
0	-2	5	-1	1
1	-1	-1	1	-2
1	-1	-1	1	-2
2	0	-3	0	3
2	0	-3	0	3
3	1	-1	-1	-2
3	1	-1	-1	-2
4	2	5	1	1
λ	1	2	$\frac{1}{2}$	$\frac{3}{2}$
Σ	12	72	6	36

x	θ_1	θ_2	θ_3	θ_4
0	-2	16	-1	1
0	-2	16	-1	1
1	-1	-11	2	-4
1	-1	-11	2	-4
2	0	-20	0	12
3	1	-11	-2	-4
3	1	-11	-2	-4
4	2	16	1	1
4	2	16	1	1
λ	1	9	$\frac{5}{6}$	$\frac{53}{12}$
Σ	20	1,908	20	212

TABLE 18 (Continued)

x	θ_1	θ_2	θ_3
0	-5	5	-5
1	-3	-1	7
2	-1	-4	4
3	1	-4	-4
4	3	-1	-7
5	5	5	5
λ	2	$\frac{3}{2}$	$\frac{5}{3}$
Σ	70	84	180

x	θ_1	θ_2	θ_3
0	-15	35	-70
0	-15	35	-70
1	-8	-24	201
2	-1	-51	102
3	6	-46	-112
4	13	-9	-186
5	20	60	135
λ	7	16	$\frac{85}{2}$
Σ	1,120	11,424	125,970

x	θ_1	θ_2	θ_3
0	-5	2	-10
1	-3	0	12
2	-1	-1	7
2	-1	-1	7
3	1	-1	-7
3	1	-1	-7
4	3	0	-12
5	5	2	10
λ	2	$\frac{1}{2}$	3
Σ	72	12	684

x	θ_1	θ_2	θ_3
0	-8	40	-151
0	-8	40	-151
1	-5	-23	396
2	-2	-54	256
3	1	-53	-124
4	4	-20	-297
4	4	-20	-297
5	7	45	184
5	7	45	184
λ	3	16	$\frac{149}{2}$
Σ	288	14,304	527,460

TABLE 18 (Continued)

x	θ_1	θ_2	θ_3
0	-3	5	-1
1	-2	0	1
2	-1	-3	1
3	0	-4	0
4	1	-3	-1
5	2	0	-1
6	3	5	1
λ	1	1	$\frac{1}{6}$
Σ	28	84	6

x	θ_1	θ_2	θ_3
0	-3	11	-1
1	-2	1	1
2	-1	-5	1
3	0	-7	0
3	0	-7	0
4	1	-5	-1
5	2	1	-1
6	3	11	1
λ	1	2	$\frac{1}{6}$
Σ	28	392	6

x	θ_1	θ_2	θ_3
0	-3	35	-14
0	-3	35	-14
1	-2	-10	29
2	-1	-37	26
3	0	-46	0
4	1	-37	-26
5	2	-10	-29
6	3	35	14
6	3	35	14
λ	1	9	$\frac{23}{6}$
Σ	46	9,954	3,818

TABLE 18 (Continued)

x	θ_1	θ_2	θ_3
0	-7	7	-7
1	-5	1	5
2	-3	-3	7
3	-1	-5	3
4	1	-5	-3
5	3	-3	-7
6	5	1	-5
7	7	7	7
λ	2	1	$\frac{2}{3}$
Σ	168	168	264

x	θ_1	θ_2
0	-28	84
0	-28	84
1	-19	-11
2	-10	-72
3	-1	-99
4	8	-92
5	17	-51
6	26	24
7	35	133
	9	17
	4,284	58,548

TABLE 18 (Contin ed)

x	θ_1	θ_2	θ_3
0	-4	28	-14
1	-3	7	7
2	-2	-8	13
3	-1	-17	9
4	0	-20	0
5	1	-17	-9
6	2	-8	-13
7	3	7	-7
8	4	28	14
λ	1	3	$\frac{5}{6}$
Σ	60	2,772	990

D. Index of Orthogonal Main-Effect Plans

The index presented in this section indicates the basic plan from which any orthogonal main-effect plan can be deduced with a minimum number of trials. Unless the experimenter must use a plan with a minimum number of trials there is usually a choice of basic plans from which an orthogonal main-effect plan can be constructed. For example, the basic plan from which the main-effect plan for the $3^4 . 2^3$ experiment can be constructed in 16 trials is basic plan 5. However a plan for this experiment can be constructed from basic plan 7 in 18 trials. The use of one basic plan over another depends on which contrasts are deemed to be most important. The orthogonal main-effect plan for the $3^4 . 2^3$ experiment in 16 trials estimates the two-level factor with an efficiency of unity and estimates the linear effect of the three-level factors with efficiency 3/4 and their quadratic effects with efficiency 9/8. The plan with 18 trials estimates the effects of the three-level factors with an efficiency of unity and the effect of each two-level factor with an efficiency of 8/9. If the effects of the three-level factors are the more important then the plan with 18 trials should be chosen and if the effects of the two-level factors are the more important then the plan with 16 trials should be chosen.

The notation used in the index requires some explanation. The plan $3^{20+n_1} . 2^{n_2}$, $\Sigma n_i = 5$ in 54 trials indicates that orthogonal main-effect plans can be constructed in 54 trials, from basic plan 22, for the following experiments:

$3^{20}.2^5$, $3^{21}.2^4$, $3^{22}.2^3$, $3^{23}.2^2$, $3^{24}.2$ and 3^{25}.

The notation is used to reduce the number of entries necessary to list all the possible plans. The plan $4^7.3^n.2^{10-3n}$, $n = 0, 1, 2$ in 32 trials indicates that the plans for the $4^7.2^{10}$, $4^7.3.2^7$ and $4^7.3^2.2^4$ experiments can be constructed in 32 trials. The plan $6^{t_1}5^{t_2}.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma t_i = 8$, $\Sigma n_i = 8-4n_1$ represents the following experiments:

$6^{t_1}5^{t_2},3^{n_2}2^{n_3}$, $\Sigma t_i = 8$, $\Sigma n_i = 8$; $6^{t_1}5^{t_2}.4.3^{n_2}.2^{n_3}$, $\Sigma t_i = 8$, $\Sigma n_i = 4$;
$6^{t_1}5^{t_2}.4^2$, $\Sigma t_i = 8$.

Plan	Number of trials	Basic plan	Page
2^3	4	1	139
2^7	8	2	139
2^{11}	12	4	140
2^{15}	16	5	141
2^{19}	20	8	142
2^{23}	24	9	143
2^{27}	28	12	146
2^{31}	32	13	147
2^{35}	36	15	149
2^{39}	40	16	150
2^{43}	44	17	151
2^{47}	48	18	152
2^{51}	52	21	155
2^{55}	56	23	157
2^{59}	60	24	158
2^{63}	64	25	159
3.2^4	8	2	139
3.2^{12}	16	5	141
3.2^{28}	32	13	147
3.2^{60}	64	25	159
$3^2.2^2$	9	3	140
$3^2.2^9$	16	5	141

Plan	Number of trials	Basic plan	Page
$3^2 . 2^{11}$	27	11	145
$3^2 . 2^{25}$	32	13	147
$3^2 . 2^{57}$	64	25	159
$3^3 . 2$	9	3	140
$3^3 . 2^6$	16	5	141
$3^3 . 2^{10}$	27	11	145
$3^3 . 2^{22}$	32	13	147
$3^3 . 2^{54}$	64	25	159
3^4	9	3	140
$3^4 . 2^3$	16	5	141
$3^4 . 2^9$	27	11	145
$3^4 . 2^{19}$	32	13	147
$3^4 . 2^{21}$	54	22	156
$3^4 . 2^{51}$	64	25	159
3^5	16	5	141
$3^5 . 2^2$	18	7	142
$3^5 . 2^8$	27	11	145
$3^5 . 2^{16}$	32	13	147
$3^5 . 2^{20}$	54	22	156
$3^5 . 2^{48}$	64	25	159

Plan	Number of trials	Basic plan	Page
$3^6 . 2$	18	7	142
$3^6 . 2^7$	27	11	145
$3^6 . 2^{13}$	32	13	147
$3^6 . 2^{19}$	54	22	156
$3^6 . 2^{45}$	64	25	159
3^7	18	7	142
$3^7 . 2^6$	27	11	145
$3^7 . 2^{10}$	32	13	147
$3^7 . 2^{18}$	54	22	156
$3^7 . 2^{42}$	64	25	159
$3^8 . 2^5$	27	11	145
$3^8 . 2^7$	32	13	147
$3^8 . 2^{17}$	54	22	156
$3^8 . 2^{39}$	64	25	159
$3^9 . 2^4$	27	11	145
$3^9 . 2^{16}$	54	22	156
$3^9 . 2^{36}$	64	25	159
$3^{10} . 2^3$	27	11	145
$3^{10} . 2^{15}$	54	22	156
$3^{10} . 2^{33}$	64	25	159

Plan	Number of trials	Basic plan	Page
$3^{11}.2^{2}$	27	11	145
$3^{11}.2^{14}$	54	22	156
$3^{11}.2^{30}$	64	25	159
$3^{12}.2$	27	11	145
$3^{12}.2^{13}$	54	22	156
$3^{12}.2^{27}$	64	25	159
$3^{12}.2^{28}$	81	26	162
3^{13}	27	11	145
$3^{13}.2^{12}$	54	22	156
$3^{13}.2^{24}$	64	25	159
$3^{13}.2^{27}$	81	26	162
$3^{14}.2^{11}$	54	22	156
$3^{14}.2^{21}$	64	25	159
$3^{14}.2^{26}$	81	26	162
$3^{15}.2^{10}$	54	22	156
$3^{15}.2^{18}$	64	25	159
$3^{15}.2^{25}$	81	26	162
$3^{16}.2^{9}$	54	22	156
$3^{16}.2^{15}$	64	25	159
$3^{16}.2^{24}$	81	26	162

Plan	Number of trials	Basic plan	Page
$3^{17}.2^{8}$	54	22	156
$3^{17}.2^{12}$	64	25	159
$3^{17}.2^{23}$	81	26	162
$3^{18}.2^{7}$	54	22	156
$3^{18}.2^{9}$	64	25	159
$3^{18}.2^{22}$	81	26	162
$3^{19}.2^{6}$	54	22	156
$3^{19}.2^{21}$	81	26	162
$3^{20+n_1}.2^{n_2}$, $\Sigma n_i = 5$	54	22	156
$3^{20+n_1}.2^{n_2}$, $\Sigma n_i = 20$	81	26	162
4.2^{4}	8	2	139
$4.3^{n}.2^{12-3n}$, $n = 0, 1, \ldots, 4$	16	5	141
$4.3^{n_1}.2^{n_2}$, $\Sigma n_i = 5$	25	10	144
$4.3^{n}.2^{28-n}$, $n = 0, 1, \ldots, 8$	32	13	147
$4.3^{n_1}.2^{n_2}$, $\Sigma n_i = 10$	50	20	154
$4.3^{n_1}.2^{3n_2}$, $\Sigma n_i = 20$	64	25	159
$4.3^{n_1}.2^{n_2}$, $\Sigma n_i = 36$	81	26	162

Plan	Number of trials	Basic plan	Page
$4^2 . 3^n . 2^{9-3n}$, $n = 0, 1, \ldots, 3$	16	5	141
$4^2 . 3^{n_1} . 2^{n_2}$, $\Sigma n_i = 4$	25	10	144
$4^2 . 3^n . 2^{25-3n}$, $n = 0, 1, \ldots, 7$	32	13	147
$4^2 . 3^{n_1} . 2^{n_2}$, $\Sigma n_i - 9$	50	20	154
$4^2 . 3^{n_1} . 2^{3n_2}$, $\Sigma n_i = 19$	64	25	159
$4^2 . 3^{n_1} . 2^{n_2}$, $\Sigma n_i = 32$	81	26	162
$4^3 . 3^n . 2^{6-3n}$, $n = 0, 1, 2$	16	5	141
$4^3 . 3^{n_1} . 2^{n_2}$, $\Sigma n_i = 3$	25	10	144
$4^3 . 3^n . 2^{22-3n}$, $n = 0, 1, \ldots, 6$	32	13	147
$4^3 . 3^{n_1} . 2^{n_2}$, $\Sigma n_i = 8$	50	20	154
$4^3 . 3^{n_1} . 2^{3n_2}$, $\Sigma n_i = 18$	64	25	159
$4^3 . 3^{n_1} . 2^{n_2}$, $\Sigma n_i = 28$	81	26	162
$4^4 . 3^n . 2^{3-3n}$, $n = 0, 1$	16	5	141
$4^4 . 3^{n_1} . 2^{n_2}$, $\Sigma n_i = 2$	25	10	144
$4^4 . 3^n . 2^{19-3n}$, $n = 0, 1, \ldots, 5$	32	13	147

Plan	Number of trials	Basic plan	Page
$4^4 . 3^{n_1} . 2^{n_2}$, $\Sigma n_i = 7$	50	20	154
$4^4 . 3^{n_1} . 2^{3n_2}$, $\Sigma n_i = 17$	64	25	159
$4^4 . 3^{n_1} . 2^{n_2}$, $\Sigma n_i = 24$	81	26	162
4^5.	16	5	141
$4^5 . 3^{n_1} . 2^{n_2}$, $\Sigma n_i = 1$	25	10	144
$4^5 . 3^n . 2^{16-3n}$, $n = 0, 1, \ldots, 4$	32	13	147
$4^5 . 3^{n_1} . 2^{n_2}$, $\Sigma n_i = 6$	50	20	154
$4^5 . 3^{n_1} . 2^{3n_2}$, $\Sigma n_i = 16$	64	25	159
$4^5 . 3^{n_1} . 2^{n_2}$, $\Sigma n_i = 20$	81	26	162
4^6.	25	10	144
$4^6 . 3^n . 2^{13-3n}$, $n = 0, 1, \ldots, 3$	32	13	147
$4^6 . 3^{n_1} . 2^{n_2}$, $\Sigma n_i = 5$	50	20	154
$4^6 . 3^{n_1} . 2^{3n_2}$, $\Sigma n_i = 15$	64	25	159
$4^6 . 3^{n_1} . 2^{n_2}$, $\Sigma n_i = 16$	81	26	162
$4^7 . 3^n . 2^{10-3n}$, $n = 0, 1, 2$	32	13	147
$4^7 . 3^{n_1} . 2^{n_2}$, $\Sigma n_i = 4$	50	20	154

Plan	Number of trials	Basic plan	Page
$4^7 . 3^{n_1} . 2^{3n_2}$, $\Sigma n_i = 14$	64	25	159
$4^8 . 3^n . 2^{7-3n}$, $n = 0, 1$	32	13	147
$4^8 . 3^{n_1} . 2^{n_2}$, $\Sigma n_i = 3$	50	20	154
$4^8 . 3^{n_1} . 2^{n_2}$, $\Sigma n_i = 13$	64	25	159
$4^9 . 2^4$	32	13	147
$4^9 . 3^{n_1} . 2^{n_2}$, $\Sigma n_i = 2$	50	20	154
$4^9 . 3^{n_1} . 2^{3n_2}$, $\Sigma n_i = 12$	64	25	159
$4^{10} . 3^{n_1} . 2^{n_2}$, $\Sigma n_i = 1$	50	20	154
$4^{10} . 3^{n_1} . 2^{3n_2}$, $\Sigma n_i = 11$	64	25	159
4^{11}	50	20	154
$4^{11+n_1} . 3^{n_2} . 2^{3n_3}$, $\Sigma n_i = 10$	64	25	159
$5 . 2^8$	16	6	141
$5 . 3^{n_1} . 2^{n_2}$, $\Sigma n_i = 9$	27	11	145
$5 . 4^{n_1} . 3^{n_2} . 2^{n_3}$, $\Sigma n_i = 5$	25	10	144

Plan	Number of trials	Basic plan	Page
$5.4^{n_1}.3^{n_2}.2^{24-3(n_1+n_2)}$, $\Sigma n_i = 0, 1, \ldots, 6$	32	14	148
$5.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma n_i = 10$	50	20	154
$5.4^{n_1}.3^{n_2}.2^{56-3(n_1+n_2)}$, $\Sigma n_i = 0, 1, \ldots, 16$	64	25	159
$5.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma n_i = 36 - 4n_1$	81	26	162
$5^2.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma n_i = 4$	25	10	144
$5^2.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma n_i = 6$	49	19	153
$5^2.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma n_i = 9$	50	20	154
$5^2.4^{n_1}.3^{n_2}.2^{49-3(n_1+n_2)}$, $\Sigma n_i = 0, 1, \ldots, 15$	64	25	159
$5^2.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma n_i = 32 - 4n_1$	81	26	162
$5^3.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma n_i = 3$	25	10	144
$5^3.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma n_i = 5$	49	19	153
$5^3.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma n_i = 8$	50	20	154
$5^3.4^{n_1}.3^{n_2}.2^{3n_3}$, $\Sigma n_i = 14$	64	25	159
$5^3.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma n_i = 28 - 4n_1$	81	26	162
$5^4.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma n_i = 2$	25	10	144

Plan	Number of trials	Basic plan	Page
$5^4 . 4^{n_1} . 3^{n_2} . 2^{n_3}$, $\Sigma n_i = 4$	49	19	153
$5^4 . 4^{n_1} . 3^{n_2} . 2^{n_3}$, $\Sigma n_i = 7$	50	20	154
$5^4 . 4^{n_1} . 3^{n_2} . 2^{35-3(n_1+n_2)}$, $\Sigma n_i = 0, 1, \ldots, 9$	64	25	159
$5^4 . 4^{n_1} . 3^{n_2} . 2^{n_3}$, $\Sigma n_i = 24-4n_1$	81	26	162
$5^5 . 4^{n_1} . 3^{n_2} . 2^{n_3}$, $\Sigma n_i = 1$	25	10	144
$5^5 . 4^{n_1} . 3^{n_2} . 2^{n_3}$, $\Sigma n_i = 3$	49	19	153
$5^5 . 4^{n_1} . 3^{n_2} . 2^{n_3}$, $\Sigma n_i = 6$	50	20	154
$5^5 . 4^{n_1} . 3^{n_2} . 2^{28-3(n_1+n_2)}$, $\Sigma n_i = 0, 1, \ldots, 8$	64	25	159
$5^5 . 4^{n_1} . 3^{n_2} . 2^{n_3}$, $\Sigma n_i = 20-4n_1$	81	26	162
$5^6 .$	25	10	144
$5^6 . 4^{n_1} . 3^{n_2} . 2^{n_3}$, $\Sigma n_i = 2$	49	19	153
$5^6 . 4^{n_1} . 3^{n_2} . 2^{n_3}$, $\Sigma n_i = 5$	50	20	154
$5^6 . 4^{n_1} . 3^{n_2} . 2^{3n_3}$, $\Sigma n_i = 7$	64	25	159
$5^6 . 4^{n_1} . 3^{n_2} . 2^{n_3}$, $\Sigma n_i = 16-4n_1$	81	26	162
$5^7 . 4^{n_1} . 3^{n_2} . 2^{n_3}$, $\Sigma n_i = 1$	49	19	153

Plan	Number of trials	Basic plan	Page
$5^7.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma n_i = 4$	50	20	154
$5^7.4^{n_1}.3^{n_2}.2^{14-3(n_1+n_2)}$, $\Sigma n_i = 0, 1, 2$	64	25	159
$5^7.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma n_i = 12-4n_1$	81	26	162
5^8	49	19	153
$5^8.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma n_i = 3$	50	20	154
$5^8.4^{n_1}.3^{n_2}.2^{7-3(n_1+n_2)}$, $\Sigma n_i = 0, 1$	64	25	159
$5^8.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma n_i = 8-4n_1$	81	26	162
$5^9.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma n_i = 2$	50	20	154
$5^9.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma n_i = 4-4n_1$	81	26	162
$5^{10}.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma n_i = 1$	50	20	154
5^{11}.	50	20	154
6.2^8	16	6	141
$6.3^{n_1}.2^{n_2}$, $\Sigma n_i = 9$	27	11	145
$6.4^{n_1}.3^{n_2}.2^{24-3(n_1+n_2)}$, $\Sigma n_i = 0, 1, \ldots, 6$	32	14	148

Plan	Number of trials	Basic plan	Page
$6.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma n_i = 7$	49	19	153
$6.4^{n_1}.3^{n_2}.2^{56-3(n_1+n_2)}$, $\Sigma n_i = 0, 1, \ldots, 16$	64	25	159
$6.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma n_i = 36-4n_1$	81	26	162
$6.5.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma n_i = 6$	49	19	153
$6.5.4^{n_1}.3^{n_2}.2^{49-3(n_1+n_2)}$, $\Sigma n_i = 0, 1, \ldots, 15$	64	25	159
$6.5.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma n_i = 32-4n_1$	81	26	162
$6^{n_1}.5^{n_2}.4^{n_3}.3^{n_4}.2^{n_5}$, $\Sigma n_i = 8$	49	19	153
$6^{t_1}.5^{t_2}.4^{n_1}.3^{n_2}.2^{3n_3}$, $\Sigma t_i = 3$, $\Sigma n_i = 14$	64	25	159
$6^{t_1}.5^{t_2}.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma t_i = 3$, $\Sigma n_i = 28-4n_1$	81	26	162
$6^{t_1}.5^{t_2}.4^{n_1}.3^{n_2}.2^{35-3(n_1+n_2)}$ $\Sigma t_i = 4$, $\Sigma n_i = 0, 1, \ldots, 9$	64	25	159
$6^{t_1}.5^{t_2}.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma t_i = 4$, $\Sigma n_i = 24-4n_1$	81	26	162
$6^{t_1}.5^{t_2}.4^{n_1}.3^{n_2}.2^{28-3(n_1+n_2)}$ $\Sigma t_i = 5$, $\Sigma n_i = 0, 1, \ldots, 8$	64	25	159
$6^{t_1}.5^{t_2}.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma t_i = 5$, $\Sigma n_i = 20-4n_1$	81	26	162
$6^{t_1}.5^{t_2}.4^{n_1}.3^{n_2}.2^{3n_3}$, $\Sigma t_i=6$, $\Sigma n_i = 7$	64	25	159

Plan	Number of trials	Basic plan	Page
$6^{t_1}.5^{t_2}.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma t_i = 6$, $\Sigma n_i = 16-4n_1$	81	26	162
$6^{t_1}.5^{t_2}.4^{n_1}.3^{n_2}.2^{14-3(n_1+n_2)}$, $\Sigma t_i = 7$, $\Sigma n_i = 0, 1, 2$	64	25	159
$6^{t_1}.5^{t_2}.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma t_i = 7$, $\Sigma n_i = 12-4n_1$	81	26	162
$6^{t_1}.5^{t_2}.4^{n_1}.3^{n_2}.2^{7-3(n_1+n_2)}$, $\Sigma t_i = 8$, $\Sigma n_i = 0, 1$	64	25	159
$6^{t_1}.5^{t_2}.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma t_i = 8$, $\Sigma n_i = 8-4n_1$	81	26	162
$6^{t_1}.5^{t_2}$, $\Sigma t_i = 9$	64	25	159
$6^{t_1}.5^{t_2}.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma t_i = 9$, $\Sigma n_i = 4-4n_1$	81	26	162
$6^{t_1}.5^{t_2}$, $\Sigma t_i = 10$	81	26	162
7.2^8	16	6	141
$7.3^{n_1}.2^{n_2}$, $\Sigma n_i = 9$	27	11	145
$7.4^{n_1}.3^{n_2}.2^{24-3(n_1+n_2)}$, $\Sigma n_i = 0, 1, \ldots, 6$	32	14	148
$7.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma n_i = 7$	49	19	153
$7.4^{n_1}.3^{n_2}.2^{56-3(n_1+n_2)}$, $\Sigma n_i = 0, 1, \ldots, 16$	64	25	159
$7.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma n_i = 36-4n_1$	81	26	162
$7.6^{n_1}.5^{n_2}.4^{n_3}.3^{n_4}.2^{n_5}$, $\Sigma n_i = 7$	49	19	153

Plan	Number of trials	Basic plan	Page
$7.6^{t_1}.5^{t_2}.4^{n_1}.3^{n_2}.2^{49-3(n_1+n_2)}$, $\Sigma t_i = 1$, $\Sigma n_i = 0, 1, \ldots, 15$	64	25	159
$7.6^{t_1}.5^{t_2}.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma t_i = 1$, $\Sigma n_i = 32-4n_1$	81	26	162
$7^{n_1}.6^{n_2}.5^{n_3}.4^{n_4}.3^{n_5}.2^{n_6}$, $\Sigma n_i = 8$	49	19	153
$7^{t_1}.6^{t_2}.5^{t_3}.4^{n_1}.3^{n_2}.2^{3n_3}$, $\Sigma t_i = 3$, $\Sigma n_i = 14$	64	25	159
$7^{t_1}.6^{t_2}.5^{t_3}.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma t_i = 3$, $\Sigma n_i = 28-4n_1$	81	26	162
$7^{t_1}.6^{t_2}.5^{t_3}.4^{n_1}.3^{n_2}.2^{35-3(n_1+n_2)}$, $\Sigma t_i = 4$, $\Sigma n_i = 0, 1, \ldots, 9$	64	25	159
$7^{t_1}.6^{t_2}.5^{t_3}.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma t_i = 4$, $\Sigma n_i = 24-4n_1$	81	26	162
$7^{t_1}.6^{t_2}.5^{t_3}.4^{n_1}.3^{n_2}.2^{28-3(n_1+n_2)}$, $\Sigma t_i = 5$, $\Sigma n_i = 0, 1, \ldots, 8$	64	25	159
$7^{t_1}.6^{t_2}.5^{t_3}.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma t_i = 5$, $\Sigma n_i = 20-4n_1$	81	26	162
$7^{t_1}.6^{t_2}.5^{t_3}.4^{n_1}.3^{n_2}.2^{3n_3}$, $\Sigma t_i = 6$, $\Sigma n_i = 7$	64	25	159
$7^{t_1}.6^{t_2}.5^{t_3}.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma t_i = 6$, $\Sigma n_i = 16-4n_1$	81	26	162
$7^{t_1}.6^{t_2}.5^{t_3}.4^{n_1}.3^{n_2}.2^{14-3(n_1+n_2)}$, $\Sigma t_i = 7$, $\Sigma n_i = 0, 1, 2$	64	25	159
$7^{t_1}.6^{t_2}.5^{t_3}.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma t_i = 7$, $\Sigma n_i = 12-4n_1$	81	26	162

Plan	Number of trials	Basic plan	Page
$7^{t_1}.6^{t_2}.5^{t_3}.4^{n_1}.3^{n_2}.2^{7-3(n_1+n_2)}$,			
$\Sigma t_i = 8,\ \Sigma n_i = 0, 1$	64	25	159
$7^{t_1}.6^{t_2}.5^{t_3}.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma t_i = 8$, $\Sigma n_i = 8-4n_1$	81	26	162
$7^{n_1}.6^{n_2}.5^{n_3}$, $\Sigma n_i = 9$	64	25	159
$7^{t_1}.6^{t_2}.5^{t_3}.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma t_i = 9$, $\Sigma n_i = 4-4n_1$	81	26	162
$7^{n_1}.6^{n_2}.5^{n_3}$, $\Sigma n_i = 10$	81	26	162
8.2^8	16	6	141
$8.3^{n_1}.2^{n_2}$, $\Sigma n_i = 9$	27	11	145
$8.4^{n_1}.3^{n_2}.2^{24-3(n_1+n_2)}$, $\Sigma n_i = 0, 1, \ldots, 6$	32	14	148
$8.4^{n_1}.3^{n_2}.2^{56-3(n_1+n_2)}$, $\Sigma n_i = 0, 1, \ldots, 16$	64	25	159
$8.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma n_i = 36-4n_1$	81	26	162
$8.7^{t_1}.6^{t_2}.5^{t_3}.4^{n_1}.3^{n_2}.2^{49-3(n_1+n_2)}$,			
$\Sigma t_i = 1,\ \Sigma n_i = 0, 1, \ldots, 15$	64	25	159
$8.7^{t_1}.6^{t_2}.5^{t_3}.4^{n_1}.3^{n_2}.2^{n_3}$,			
$\Sigma t_i = 1,\ \Sigma n_i = 32-4n_1$	81	26	162
$8^{t_1}.7^{t_2}.6^{t_3}.5^{t_4}.4^{n_1}.3^{n_2}.2^{3n_3}$, $\Sigma t_i = 3$, $\Sigma n_i = 14$	64	25	159

Plan	Number of trials	Basic plan	Page
$8^{t_1}.7^{t_2}.6^{t_3}.5^{t_4}.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma t_i = 3$, $\Sigma n_i = 28-4n_1$	81	26	162
$8^{t_1}.7^{t_2}.6^{t_3}.5^{t_4}.4^{n_1}.3^{n_2}.2^{35-3(n_1+n_2)}$, $\Sigma t_i = 4$, $\Sigma n_i = 0, 1, \ldots, 9$	64	25	159
$8^{t_1}.7^{t_2}.6^{t_3}.5^{t_4}.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma t_i = 4$, $\Sigma n_i = 24-4n_1$	81	26	162
$8^{t_1}.7^{t_2}.6^{t_3}.5^{t_4}.4^{n_1}.3^{n_2}.2^{28-3(n_1+n_2)}$, $\Sigma t_i = 5$, $\Sigma n_i = 0, 1, \ldots, 8$	64	25	159
$8^{t_1}.7^{t_2}.6^{t_3}.5^{t_4}.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma t_i = 5$, $\Sigma n_i = 20-4n_1$	81	26	162
$8^{t_1}.7^{t_2}.6^{t_3}.5^{t_4}.4^{n_1}.3^{n_2}.2^{3n_3}$, $\Sigma t_i = 6$, $\Sigma n_i = 7$	64	25	159
$8^{t_1}.7^{t_2}.6^{t_3}.5^{t_4}.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma t_i = 6$, $\Sigma n_i = 16-4n_1$	81	26	162
$8^{t_1}.7^{t_2}.6^{t_3}.5^{t_4}.4^{n_1}.3^{n_2}.2^{14-3(n_1+n_2)}$, $\Sigma t_i = 7$, $\Sigma n_i = 0, 1, 2$	64	25	159
$8^{t_1}.7^{t_2}.6^{t_3}.5^{t_4}.4^{n_1}.3^{n_2}.2^{n_3}$, $\Sigma t_i = 7$, $\Sigma n_i = 12-4n_1$	81	26	162

Plan	Number of trials	Basic plan	Page
$8^{t_1}.7^{t_2}.6^{t_3}.5^{t_4}.4^{n_1}.3^{n_2}.2^{7-3(n_1+n_2)}$,			
$\Sigma t_i = 8$, $\Sigma n_i = 0, 1$	64	25	159
$8^{t_1}.7^{t_2}.6^{t_3}.5^{t_4}\,4^{n_1}.3^{n_2}.2^{n_3}$,			
$\Sigma t_i = 8$, $\Sigma n_i = 8-4n_1$	81	26	162
$8^{n_1}.7^{n_2}.6^{n_3}.5^{n_4}$, $\Sigma n_i = 9$	64	25	159
$8^{t_1}.7^{t_2}.6^{t_3}.5^{t_4}.4^{n_1}.3^{n_2}.2^{n_3}$,			
$\Sigma t_i = 9$, $\Sigma n_i = 4-4n_1$	81	26	162
$8^{n_1}.7^{n_2}.6^{n_3}.5^{n_4}$, $\Sigma n_i = 10$	81	26	162
$9.3^{n_1}.2^{n_2}$, $\Sigma n_i = 9$	27	11	145
$9^{n_1}.8^{n_2}.7^{n_3}.6^{n_4}.5^{n_5}.4^{n_6}.3^{n_7}.2^{n_8}$, $\Sigma n_i = 10$	81	26	162

F. Basic Orthogonal Main-Effect Plans

BASIC PLAN 1: 2^3; 4 trials

```
123

000
011
101
110
```

BASIC PLAN 2: 4; 3; 2^7; 8 trials

```
*  *  1234567

0  0  0000000
0  0  0001111
1  1  0110011
1  1  0111100
2  2  1010101
2  2  1011010
3  1  1100110
3  1  1101001

     *-1,2,3
```

BASIC PLAN 3: 3^4; 2^4; 9 trials

1234	1234
0000	0000
0112	0110
0221	0001
1011	1011
1120	1100
1202	1000
2022	0000
2101	0101
2210	0010

BASIC PLAN 4: 2^{11}; 12 trials

00000 12345	000011 678901
00000	000000
11011	100010
01101	110001
10110	111000
01011	011100
00101	101110
00010	110111
10001	011011
11000	101101
11100	010110
01110	001011
10111	000101

BASIC PLAN 5: 4^5 ; 3^5 ; 2^{15} ; 16 trials

12345	12345	00000	00001	11111
*****	*****	12345	67890	12345
00000	00000	00000	00000	00000
01123	01121	00001	10111	01110
02231	02211	00010	11011	10011
03312	01112	00011	01100	11101
10111	10111	01100	00110	11011
11032	11012	01101	10001	10101
12320	12120	01110	11101	01000
13203	11201	01111	01010	00110
20222	20222	10100	01011	01101
21301	21101	10101	11100	00011
22013	22011	10110	10000	11110
23130	21110	10111	00111	10000
30333	10111	11000	01101	10110
31210	11210	11001	11010	11000
32102	12102	11010	10110	00101
33021	11021	11011	00001	01011

1-000	2-000	3-000	4-111	5-111
*-123	*-456	*-789	*-012	*-345

BASIC PLAN 6: 8 ; 7 ; 6 ; 5 ; 2^8 ; 16 trials

1	1	1	1	23456789
0	0	0	0	00000000
0	0	0	0	11111111
1	1	1	1	00001111
1	1	1	1	11110000
2	2	2	2	00110011
2	2	2	2	11001100
3	3	3	3	00111100
3	3	3	3	11000011
4	4	4	4	01010101
4	4	4	4	10101010
5	5	5	1	01011010
5	5	5	1	10100101
6	6	2	2	01100110
6	6	2	2	10011001
7	3	3	3	01101001
7	3	3	3	10010110

BASIC PLAN 7: 3^7 ; 2^7 ; 18 trials

1234567	1234567
0000000	0000000
0112111	0110111
0221222	0001000
1011120	1011100
1120201	1100001
1202012	1000010
2022102	0000100
2101210	0101010
2210021	0010001
0021011	0001011
0100122	0100100
0212200	0010000
1002221	1000001
1111002	1111000
1220110	1000110
2010212	0010010
2122020	0100000
2201101	0001101

BASIC PLAN 8: 2^{19} ; 20 trials

00000	00001	11111	1111
12345	67890	12345	6789
00000	00000	00000	0000
11001	11101	01000	0110
01100	11110	10100	0011
10110	01111	01010	0001
11011	00111	10101	0000
01101	10011	11010	1000
00110	11001	11101	0100
00011	01100	11110	1010
00001	10110	01111	0101
10000	11011	00111	1010
01000	01101	10011	1101
10100	00110	11001	1110
01010	00011	01100	1111
10101	00001	10110	0111
11010	10000	11011	0011
11101	01000	01101	1011
11110	10100	00110	1110
01111	01010	00011	0110
00111	10101	00001	1011
10011	11010	10000	1101

BASIC PLAN 9: 2^{23}; 24 trials

00000	00001	11111	11112	222
12345	67890	12345	67890	123
00000	00000	00000	00000	000
11111	01011	00110	01010	000
01111	10101	10011	00101	000
00111	11010	11001	10010	100
00011	11101	01100	11001	010
00001	11110	10110	01100	101
10000	11111	01011	00110	010
01000	01111	10101	10011	001
10100	00111	11010	11001	100
01010	00011	11101	01100	110
00101	00001	11110	10110	011
10010	10000	11111	01011	001
11001	01000	01111	10101	100
01100	10100	00111	11010	110
00110	01010	00011	11101	011
10011	00101	00001	11110	101
11001	10010	10000	11111	010
01100	11001	01000	01111	101
10110	01100	10100	00111	110
01011	00110	01010	00011	111
10101	10011	00101	00001	111
11010	11001	10010	10000	111
11101	01100	11001	01000	011
11110	10110	01100	10100	001

BASIC PLAN 10: 5^6; 4^6; 3^6; 2^6; 25 trials

123456	123456	123456	123456
000000	000000	000000	000000
011234	011230	011220	011110
022413	022013	022012	011011
033142	033102	022102	011101
044321	000321	000221	000111
101111	101111	101111	101111
112340	112300	112200	111100
123024	123020	122020	111010
134203	130203	120202	110101
140432	100032	100022	100011
202222	202222	202222	101111
213401	213001	212001	111001
224130	220130	220120	110110
230314	230310	220210	110110
241043	201003	201002	101001
303333	303333	202222	101111
314012	310012	210012	110011
320241	320201	220201	110101
331420	331020	221020	111010
342104	302100	202100	101100
404444	000000	000000	000000
410123	010123	010122	010111
421302	021302	021202	011101
432031	032031	022021	011011
443210	003210	002210	001110

BASIC PLAN 11: 9 ; 8 ; 7 ; 6 ; 5 ; 4 ; 3^{13} ; 2^{13} ; 27 trials

						00000	00001	111	00000	00001	111
*	*	*	*	*	*	12345	67890	123	12345	67890	123
0	0	0	0	0	0	00000	00000	000	00000	00000	000
0	0	0	0	0	0	00001	12121	212	00001	10101	010
0	0	0	0	0	0	00002	21212	121	00000	01010	101
1	1	1	1	1	1	01120	00111	122	01100	00111	100
1	1	1	1	1	1	01121	12202	001	01101	10000	001
1	1	1	1	1	1	01122	21020	210	01100	01000	010
2	2	2	2	2	2	02210	00222	211	00010	00000	011
2	2	2	2	2	2	02211	12010	120	00011	10010	100
2	2	2	2	2	2	02212	21101	00[illegible]	00010	01101	000
3	3	3	3	1	1	10110	11001	111	10110	11001	111
3	3	3	3	1	1	10111	20122	020	10111	00100	000
3	3	3	3	1	1	10112	02210	202	10110	00010	000
4	4	4	4	3	3	11200	11112	200	11000	11110	000
4	4	4	4	3	3	11201	20200	112	11001	00000	110
4	4	4	4	3	3	11202	02021	021	11000	00001	001
5	5	5	4	3	3	12020	11220	022	10000	11000	0[illegible]
5	5	5	4	3	3	12021	20011	201	10001	00011	001
5	5	5	4	3	3	12022	02102	110	10000	00100	110
6	6	6	5	4	2	20220	22002	222	00000	00000	000
6	6	6	5	4	2	20221	01120	101	00001	01100	101
6	6	6	5	4	2	20222	10211	10	00000	10011	010
7	7	6	5	4	2	21010	22110	011	01010	00110	011
7	7	6	5	4	2	21011	01201	220	01011	01001	000
7	7	6	5	4	2	21012	10022	102	01010	10000	100
8	0	0	0	0	0	22100	22221	100	00100	00001	100
8	0	0	0	0	0	22101	01012	012	00101	01010	010
8	0	0	0	0	0	22102	10100	221	00100	10100	001

*-1,2,3,4

BASIC PLAN 12: 2^{27}; 28 trials

00000	00001	11111	11112	22222	22
12345	67890	12345	67890	12345	67
00000	00000	00000	00000	00000	00
10111	10000	10001	00111	01011	01
11011	10000	01100	10001	11101	10
01111	10001	00010	01010	10110	11
00010	11110	01010	00110	11101	01
00011	01111	00001	10011	00111	10
00001	11110	10100	01001	11010	11
11100	01010	01001	01010	11011	10
11100	01101	00100	00111	01100	11
11100	00110	10010	10001	10111	01
10001	00101	01011	01101	11110	00
10111	01010	00101	11100	10100	01
11001	11100	00110	11110	00011	00
01110	10110	00011	11101	01000	10
10110	11101	11000	10100	10010	10
11011	00111	11000	11010	01000	01
01101	11011	11000	01101	00101	00
00110	01000	11110	11011	01110	00
00101	00011	01110	10100	01011	11
10000	11001	10011	11000	01101	11
01010	00100	11101	01100	00111	11
00100	10101	01101	11011	10001	01
10010	00011	10110	01111	10001	10
01001	01000	11011	10111	10000	11
11010	11011	01111	00001	00010	01
01111	01101	10111	00000	11001	00
01000	10011	10101	10110	11110	00
10101	10110	11111	00010	00100	10

BASIC PLAN 13: 4^9; 3^9; 2^{31}; 32 trials

123456789	123456789	00000	00001	11111	11112	22222	22	2233
*********	*********	12345	67890	12345	67890	12345	67	8901
000000000	000000000	00000	00000	00000	00000	00000	00	0000
011231111	011211111	00001	10111	01110	01101	10110	11	0000
022312222	022112222	00010	11011	10011	10110	11011	01	0000
033123333	011121111	00011	01100	11101	11011	01101	10	0000
101111032	101111012	01100	00110	11011	01100	01101	01	0011
110320123	110120121	01101	10001	10101	00001	11011	10	0011
123203210	121201210	01110	11101	01000	11010	10110	00	0011
132032301	112012101	01111	01010	00110	10111	00000	11	0011
202223102	202221102	10100	01011	01101	11001	10001	01	0101
213012013	211012011	10101	11100	00011	10100	00111	10	0101
220131320	220111120	10110	10000	11110	01111	01010	00	0101
231300231	211100211	10111	00111	10000	00010	11100	11	0101
303332130	101112110	11000	01101	10110	10101	11100	00	0110
312103021	112101021	11001	11010	11000	11000	01010	11	0110
321020312	121020112	11010	10110	00101	00011	00111	01	0110
330211203	110211201	11011	00001	01011	01110	10001	10	0110
002130213	002110211	00000	01010	11110	00010	10111	10	1111
013301302	011101102	00001	11101	10000	01111	00001	01	1111
020222031	020222011	00010	10001	01101	10100	01100	11	1111
031013120	011011120	00011	00110	00011	11001	11010	00	1111
103021221	101021221	01100	01100	00101	01110	11010	11	1100
112210330	112210110	01101	11011	01011	00011	01100	00	1100
121333003	121111001	01110	10111	10110	11000	00001	10	1100
130102112	110102112	01111	00000	11000	10101	10111	01	1100
200313311	200111111	10100	00001	10011	11011	00110	11	1010
211122200	211122200	10101	10110	11101	10110	10000	00	1010
222001133	222001111	10110	11010	00000	01101	11101	10	1010
233230022	211210022	10111	01101	01110	00000	01011	01	1010
301202323	101202121	11000	00111	01000	10111	01011	10	1001
310033232	110011212	11001	10000	00110	11010	11101	01	1001
323110101	121110101	11010	11100	11011	00001	10000	11	1001
332321010	112121010	11011	01011	10101	01100	00110	00	1001

1-000	2-000	3-000	4-111	5-111	6-111	7-122	8-222	9-222
*-123	*-456	*-789	*-012	*-345	*-678	*-901	*-234	*-567

BASIC PLAN 14: 8 ; 7 ; 6 ; 5 ; 4^6 ; 3^6 ; 2^{24} ; 32 trials

0	0	0	0	234567	234567	00000	00011	11111	111	222222
1	1	1	1	******	******	23456	78901	23456	789	012345
0	0	0	0	000000	000000	00000	00000	00000	000	000000
1	1	1	1	010123	010121	00001	10000	11101	110	001111
3	3	3	3	001212	001212	00000	00111	01011	101	011110
2	2	2	2	011331	011111	00001	10111	10110	011	010001
1	1	1	1	102011	102011	01100	01010	00011	011	110101
0	0	0	0	112132	112112	01101	11010	11110	101	111010
2	2	2	2	103203	101201	01100	01101	01000	110	101011
3	3	3	3	113320	111120	01101	11101	10101	000	100100
2	2	2	2	220022	220022	10110	10000	00101	101	111100
3	3	3	3	230101	210101	10111	00000	11000	011	110011
1	1	1	1	221230	221210	10110	10111	01110	000	100010
0	0	0	0	231313	211111	10111	00111	10011	110	101101
3	3	3	3	322033	122011	11010	11010	00110	110	001001
2	2	2	2	332110	112110	11011	01010	11011	000	000110
0	0	0	0	323221	121221	11010	11101	01101	011	010111
1	1	1	1	333302	111102	11011	01101	10000	101	011000
4	4	4	4	121101	121101	01110	10110	11000	011	001100
5	5	5	1	131022	111022	01111	00110	00101	101	000011
7	3	3	3	120313	120111	01110	10001	10011	110	010010
6	6	2	2	130230	110210	01111	00001	01110	000	011101
5	5	5	1	023110	021110	00010	11100	11011	000	111001
4	4	4	4	033033	011011	00011	01100	00110	110	110110
6	6	2	2	022302	022102	00010	11011	10000	101	100111
7	3	3	3	032221	012221	00011	01011	01101	011	101000
6	6	2	2	301123	101121	11000	00110	11101	110	110000
7	3	3	3	311000	111000	11001	10110	00000	000	111111
5	5	5	1	300331	100111	11000	00001	10110	011	101110
4	4	4	4	310212	110212	11001	10001	01011	101	100001
7	3	3	3	203132	201112	10100	01100	11110	101	000101
6	6	2	2	213011	211011	10101	11100	00011	011	001010
4	4	4	4	202320	202120	10100	01011	10101	000	011011
5	5	5	1	212203	212201	10101	11011	01000	110	010100

2-000 3-000 4-001 5-111 6-111 7-111

*-234 *-567 *-890 *-123 *-456 *-789

BASIC PLAN 15: 2^{35}; 36 trials

00000	00001	11111	11112	22222	22223	33333
12345	67890	12345	67890	12345	67890	12345
00000	00000	00000	00000	00000	00000	00000
01011	10001	11110	11100	10000	10101	10010
00101	11000	11111	01110	01000	01010	11001
10010	11100	01111	10111	00100	00101	01100
01001	01110	00111	11011	10010	00010	10110
00100	10111	00011	11101	11001	00001	01011
10010	01011	10001	11110	11100	10000	10101
11001	00101	11000	11111	01110	01000	01010
01100	10010	11100	01111	10111	00100	00101
10110	01001	01110	00111	11011	10010	00010
01011	00100	10111	00011	11101	11001	00001
10101	10010	01011	10001	11110	11100	10000
01010	11001	00101	11000	11111	01110	01000
00101	01100	10010	11100	01111	10111	00100
00010	10110	01001	01110	00111	11011	10010
00001	01011	00100	10111	00011	11101	11001
10000	10101	10010	01011	10001	11110	11100
01000	01010	11001	00101	11000	11111	01110
00100	00101	01100	10010	11100	01111	10111
10010	00010	10110	01001	01110	00111	11011
11001	00001	01011	00100	10111	00011	11101
11100	10000	10101	10010	01011	10001	11110
01110	01000	01010	11001	00101	11000	11111
10111	00100	00101	01100	10010	11100	01111
11011	10010	00010	10110	01001	01110	00111
11101	11001	00001	01011	00100	10111	00011
11110	11100	10000	10101	10010	01011	10001
11111	01110	01000	01010	11001	00101	11000
01111	10111	00100	00101	01100	10010	11100
00111	11011	10010	00010	10110	01001	01110
00011	11101	11001	00001	01011	00100	10111
10001	11110	11100	10000	10101	10010	01011
11000	11111	01110	01000	01010	11001	00101
11100	01111	10111	00100	00101	01100	10010
01110	00111	11011	10010	00010	10110	01001
10111	00011	11101	11001	00001	01011	00100

BASIC PLAN 16: 2^{39}, 40 trials

00000	00001	11111	11112	22222	22223	33333	3333
12345	67890	12345	67890	12345	67890	12345	6789
00000	00000	00000	00000	00000	00000	00000	0000
11001	11101	01000	01100	11001	11101	01000	0110
01100	11110	10100	00110	01100	11110	10100	0011
10110	01111	01010	00010	10110	01111	01010	0001
11011	00111	10101	00000	11011	00111	10101	0000
01101	10011	11010	10000	01101	10011	11010	1000
00110	11001	11101	01000	00110	11001	11101	0100
00011	01100	11110	10100	00011	01100	11110	1010
00001	10110	01111	01010	00001	10110	01111	0101
10000	11011	00111	10100	10000	11011	00111	1010
01000	01101	10011	11010	01000	01101	10011	1101
10100	00110	11001	11100	10100	00110	11001	1110
01010	00011	01100	11110	01010	00011	01100	1111
10101	00001	10110	01110	10101	00001	10110	0111
11010	10000	11011	00110	11010	10000	11011	0011
11101	01000	01101	10010	11101	01000	01101	1001
11110	10100	00110	11000	11110	10100	00110	1100
01111	01010	00011	01100	01111	01010	00011	0110
00111	10101	00001	10110	00111	10101	00001	1011
10011	11010	10000	11010	10011	11010	10000	1101
00000	00000	00000	00001	11111	11111	11111	1111
11001	11101	01000	01101	00110	00010	10111	1001
01100	11110	10100	00111	10011	00001	01011	1100
10110	01111	01010	00011	01001	10000	10101	1110
11011	00111	10101	00001	00100	11000	01010	1111
01101	10011	11010	10001	10010	01100	00101	0111
00110	11001	11101	01001	11001	00110	00010	1011
00011	01100	11110	10101	11100	10011	00001	0101
00001	10110	01111	01011	11110	01001	10000	1010
10000	11011	00111	10101	01111	00100	11000	0101
01000	01101	10011	11011	10111	10010	01100	0010
10100	00110	11001	11101	01011	11001	00110	0001
01010	00011	01100	11111	10101	11100	10011	0000
10101	00001	10110	01111	01010	11110	01001	1000
11010	10000	11011	00111	00101	01111	00100	1100
11101	01000	01101	10011	00010	10111	10010	0110
11110	10100	00110	11001	00001	01011	11001	0011
01111	01010	00011	01101	10000	10101	11100	1001
00111	10101	00001	10111	11000	01010	11110	0100
10011	11010	10000	11011	01100	00101	01111	0010

BASIC PLAN 17: 2^{43}; 44 trials

00000	00001	11111	11112	22222	22223	33333	33334	444
12345	67890	12345	67890	12345	67890	12345	67890	123
00000	00000	00000	00000	00000	00000	00000	00000	000
11001	01001	11011	11100	01011	10000	01000	11010	110
01100	10100	11101	11110	00101	11000	00100	01101	011
10110	01010	01110	11111	00010	11100	00010	00110	101
11011	00101	00111	01111	10001	01110	00001	00011	010
01101	10010	10011	10111	11000	10111	00000	10001	101
10110	11001	01001	11011	11100	01011	10000	01000	110
01011	01100	10100	11101	11110	00101	11000	00100	011
10101	10110	01010	11110	11111	00010	11100	00010	001
11010	11011	00101	00111	01111	10001	01110	00001	000
01101	01101	10010	10011	10111	11000	10111	00000	100
00110	10110	11001	01001	11011	11100	01011	10000	010
00011	01011	01100	10100	11101	11110	00101	11000	001
10001	10101	10110	01010	01110	11111	00010	11100	000
01000	11010	11011	00101	00111	01111	10001	01110	000
00100	01101	01101	10010	10011	10111	11000	10111	000
00010	00110	10110	11001	01001	11011	11100	01011	100
00001	00011	01011	01100	10100	11101	11110	00101	110
00000	10001	10101	10110	01010	01110	11111	00010	111
10000	01000	11010	11011	00101	00111	01111	10001	011
11000	00100	01101	01101	10010	10011	10111	11000	101
11100	00010	00110	10110	11001	01001	11011	11100	010
01110	00001	00011	01011	01100	10100	11101	11110	001
10111	00000	10001	10101	10110	01010	01110	11111	000
01011	10000	01000	11010	11011	00101	00111	01111	100
00101	11000	00100	01101	01101	10010	10011	10111	110
00010	11100	00010	00110	10110	11001	01001	11011	111
10001	01110	00001	00011	01011	01100	10100	11101	111
11000	10111	00000	10001	10101	10110	01010	01110	111
11100	01011	10000	01000	11010	11011	00101	00111	011
11110	00101	11000	00100	01101	01101	10010	10011	101
11111	00010	11100	00010	00110	10110	11001	01001	110
01111	10001	01110	00001	00011	01011	01100	10100	111
10111	11000	10111	00000	10001	10101	10110	01010	011
11011	11100	01011	10000	01000	11010	11011	00101	001
11101	11110	00101	11000	00100	01101	01101	10010	100
01110	11111	00010	11100	00010	00110	10110	11001	010
00111	01111	10001	01110	00001	00011	01011	01100	101
10011	10111	11000	10111	00000	10001	10101	10110	010
01001	11011	11100	01011	10000	01000	11010	11011	001
10100	11101	11110	00101	11000	00100	01101	01101	100
01010	01110	11111	00010	11100	00010	00110	10110	110
00101	00111	01111	10001	01110	00001	00011	01011	011
10010	10011	10111	11000	10111	00000	10001	10101	101

00000	00001	11111	11112	22222	22223	33333	33334	44444	44
12345	67890	12345	67890	12345	67890	12345	67890	12345	67
00000	00000	00000	00000	00000	00000	00000	00000	00000	00
11111	01111	00101	01110	01001	10110	00101	01100	00100	00
01111	10111	10010	10111	00100	11011	00010	10110	00010	00
00111	11011	11001	01011	10010	01101	10001	01011	00001	00
00011	11101	11100	10101	11001	00110	11000	10101	10000	10
00001	11110	11110	01010	11100	10011	01100	01010	11000	01
10000	11111	01111	00101	01110	01001	10110	00101	01100	00
01000	01111	10111	10010	10111	00100	11011	00010	10110	00
00100	00111	11011	11001	01011	10010	01101	10001	01011	00
00010	00011	11101	11100	10101	11001	00110	11000	10101	10
00001	00001	11110	11110	01010	11100	10011	01100	01010	11
10000	10000	11111	01111	00101	01110	01001	10110	00101	01
11000	01000	01111	10111	10010	10111	00100	11011	00010	10
01100	00100	00111	11011	11001	01011	10010	01101	10001	01
10110	00010	00011	11101	11100	10101	11001	00110	11000	10
01011	00001	00001	11110	11110	01010	11100	10011	01100	01
10101	10000	10000	11111	01111	00101	01110	01001	10110	00
01010	11000	01000	01111	10111	10010	10111	00100	11011	00
00101	01100	00100	00111	11011	11001	01011	10010	01101	10
00010	10110	00010	00011	11101	11100	10101	11001	00110	11
10001	01011	00001	00001	11110	11110	01010	11100	10011	01
11000	10101	10000	10000	11111	01111	00101	01110	01001	10
01100	01010	11000	01000	01111	10111	10010	10111	00100	11
10110	00101	01100	00100	00111	11011	11001	01011	10010	01
11011	00010	10110	00010	00011	11101	11100	10101	11001	00
01101	10001	01011	00001	00001	11110	11110	01010	11100	10
00110	11000	10101	10000	10000	11111	01111	00101	01110	01
10011	01100	01010	11000	01000	01111	10111	10010	10111	00
01001	10110	00101	01100	00100	00111	11011	11001	01011	10
00100	11011	00010	10110	00010	00011	11101	11100	10101	11
10010	01101	10001	01011	00001	00001	11110	11110	01010	11
11001	00110	11000	10101	10000	10000	11111	01111	00101	01
11100	10011	01100	01010	11000	01000	01111	10111	10010	10
01110	01001	10110	00101	01100	00100	00111	11011	11001	01
10111	00100	11011	00010	10110	00010	00011	11101	11100	10
01011	10010	01101	10001	01011	00001	00001	11110	11110	01
10101	11001	00110	11000	10101	10000	10000	11111	01111	00
01010	11100	10011	01100	01010	11000	01000	01111	10111	10
00101	01110	01001	10110	00101	01100	00100	00111	11011	11
10010	10111	00100	11011	00010	10110	00010	00011	11101	11
11001	01011	10010	01101	10001	01011	00001	00001	11110	11
11100	10101	11001	00110	11000	10101	10000	10000	11111	01
11110	01010	11100	10011	01100	01010	11000	01000	01111	10
01111	00101	01110	01001	10110	00101	01100	00100	00111	11
10111	10010	10111	00100	11011	00010	10110	00010	00011	11
11011	11001	01011	10010	01101	10001	01011	00001	00001	11
11101	11100	10101	11001	00110	11000	10101	10000	10000	11
11110	11110	01010	11100	10011	01100	01010	11000	01000	01

12345678	12345678	12345678	12345678	12345678	12345678
00000000	00000000	00000000	00000000	00000000	00000000
01123456	01123450	01123440	01122330	01122110	01100110
02246135	02240135	02240134	02230123	02210121	00010101
03362514	03302514	03302414	02202313	02202111	00000111
04415263	04415203	04414203	03313202	01111202	01111000
05531642	05531042	04431042	03321032	01121012	01101010
06654321	00054321	00044321	00033221	00011221	00011001
10111111	10111111	10111111	10111111	10111111	10111111
11234560	11234500	11234400	11223300	11221100	11001100
12350246	12350240	12340240	12230230	12210210	10010010
13403625	13403025	13403024	12302023	12102021	10100001
14526304	14520304	144[illegible]0304	13320203	11120201	11100001
15642053	15042053	14042043	13032032	11012012	11010010
16065432	10005432	10004432	10003322	10001122	10001100
20222222	20222222	20222222	20222222	20222222	00000000
21345601	21345001	21344001	21233001	21211001	01011001
22461350	22401350	22401340	22301230	22101210	00101010
23514036	23514030	23414030	22313020	22111020	00111000
24630415	24030415	24030414	23020313	21020111	01000111
25053164	25053104	24043104	23032103	21012101	01010101
26106543	20100543	20100443	20100332	20100112	00100110
30333333	30333333	30333333	20222222	20222222	00000000
31456012	31450012	31440012	21330012	21110012	011100[illegible]0
32502461	32502401	32402401	22302301	22102101	00100101
33625140	33025140	33024140	22023130	22021110	00001110
34041526	34041520	34041420	23031320	21011120	01011100
35164205	35104205	34104204	23103203	21101201	01101001
36210654	30210054	30210044	20210023	20210011	00010011
40444444	40444444	40444444	30333333	10111111	10111111
41560123	41500123	41400123	31300122	11100122	11100100
42613502	42013502	42013402	32012302	12012102	10010100
4302[illegible]251	43030251	43030241	32020231	12020211	10000011
44152630	44152030	44142030	33132020	11112020	11110000
45205316	45205310	44204310	33203210	11201210	11001010
46321065	40321005	40321004	30221003	10221001	10001001
50555555	50555555	40444444	30333333	10111111	10111111
51601234	51001234	41001234	31001223	11001221	11001001
52024613	52024013	42024013	32023012	12021012	10001010
53140362	53140302	43140302	32130202	12110202	10110000
54263041	54203041	44203041	33202031	11202011	11000011
55316420	55310420	44310420	33210320	11210120	11010100
56432106	50432100	40432100	30322100	10122100	10100100
60666666	00000000	00000000	00000000	00000000	00000000
61012345	01012345	01012344	01012233	01012211	01010011
62135024	02135024	02134024	02123023	02121021	00101001
63251403	03251403	03241403	02231302	02211102	00011100
64304152	04304152	04304142	03203132	01201112	01001110
65420531	05420531	04420431	03320321	01120121	01100101
66543210	00543210	00443210	00332210	00112210	00110010

00000000011	00000000011	00000000011	00000000011
12345678901	12345678901	12345678901	12345678901
00000000000	00000000000	00000000000	00000000000
01123411111	01123011111	01122011111	01111011111
02241322222	02201322222	02201222222	01101111111
03314233333	03310233333	02210222222	01110111111
04432144444	00032100000	00022100000	00011100000
10111112340	10111112300	10111112200	10111111100
11234023401	11230023001	11220022001	11110011001
12302434012	12302030012	12202020012	11101010011
13420340123	13020300123	12020200122	11010100111
14043201234	10003201230	10002201220	10001101110
20222241302	20222201302	20222201202	10111101101
21340102413	21300102013	21200102012	11100101011
22413013024	22013013020	22012012020	11011011010
23031424130	23031020130	22021020120	11011010110
24104330241	20100330201	20100220201	10100110101
30333342031	30333302031	20222202021	10111101011
31401203142	31001203102	21001202102	11001101101
32024114203	32020110203	22020110202	11010110101
33142020314	33102020310	22102020210	11101010110
34210431420	30210031020	20210021020	10110011010
40444410432	00000010032	00000010022	00000010011
41012321043	01012321003	01012221002	01011111001
42130232104	02130232100	02120222100	01110111100
43203143210	03203103210	02202102210	01101101110
44321004321	00321000321	00221000221	00111000111
00132403223	00132003223	00122002222	00111001111
01200314334	01200310330	01200210220	01100110110
02323220440	02323220000	02222220000	01111110000
03441131001	03001131001	02001121001	01001111001
04014042112	00010002112	00010002112	00010001111
10243034143	10203030103	10202020102	10101010101
11311440204	11311000200	11211000200	11111000100
12434301310	12030301310	12020201210	11010101110
13002212421	13002212021	12002212021	11001111011
14120123032	10120123032	10120122022	10110111011
20304121124	20300121120	20200121120	10100111110
21422032230	21022032230	21022022220	11011011110
22040443341	22000003301	22000002201	11000001101
23113304402	23113300002	22112200002	11111100001
24231210013	20231210013	20221210012	10111110011
30410224211	30010220211	20010220211	10010110111
31033130322	31033130322	21022120222	11011110111
32101041433	32101001033	22101001022	11101001011
33224402044	33220002000	22220002000	11110001000
34342313100	30302313100	20202212100	10101111100
40021333414	00021333010	00021222010	00011111010
41144244020	01100200020	01100200020	01100100010
42212100131	02212100131	02212100121	01111100111
43330011242	03330011202	02220011202	01110011101
44403422303	00003022303	00002022202	00001011101

00000	00001	11111	11112	22222	22223	33333	33334	44444	444455
12345	67890	12345	67890	12345	67890	12345	67890	12345	678901
00000	00000	00000	00000	00000	00000	00000	00000	00000	000000
11010	10101	01010	10101	01010	10101	01010	10101	01010	101010
10100	00000	01100	00111	11111	00110	01100	11001	11111	110000
01101	01010	11001	01101	01010	01100	11001	10011	01010	100101
10001	00000	01111	00001	10011	11001	11111	00110	00011	111100
00111	01010	11010	01011	00110	10011	01010	01100	10110	101001
10000	01000	01111	11000	01100	11110	00011	11001	10000	111111
00101	11010	11010	10010	11001	10100	10110	10011	00101	101010
10000	00010	00011	11110	00011	00111	11100	11110	01100	001111
00101	01110	10110	10100	10110	01101	01001	10100	11001	011010
10000	00000	10000	11111	11100	11001	10011	00111	11111	000011
00101	01011	10101	10101	01001	10011	00110	01101	01010	010110
11111	11000	00100	00000	01100	00111	11111	00110	01100	110011
01010	10010	11101	01010	11001	01101	01010	01100	11001	100110
10011	11110	00001	00000	01111	00001	10011	11001	11111	001100
00110	10100	10111	01010	11010	01011	00110	10011	01010	011001
10000	11111	10000	01000	01111	11000	01100	11110	00011	110011
00101	10101	00101	11010	11010	10010	11001	10100	10110	100110
11100	00111	10000	00010	00011	11110	00011	00111	11100	111100
01001	01101	00101	01110	10110	10100	10110	01101	01001	101001
11111	00001	10000	00000	10000	11111	11100	11001	10011	001111
01010	01011	00101	01011	10101	10101	01001	10011	00110	011010
11100	11001	11111	11000	00100	00000	01100	00111	11111	001100
01001	10011	01010	10010	11101	01010	11001	01101	01010	011001
11111	00110	00011	11110	00001	00000	01111	00001	10011	110011
01010	01100	10110	10100	10111	01010	11010	01011	00110	100110
10011	11001	10000	11111	10000	01000	01111	11000	01100	111100
00110	10011	00101	10101	00101	11010	11010	10010	11001	101001
11100	11110	[illegible]	00111	10000	00010	00011	11110	00011	001111
01001	10100	11001	01101	00101	01110	10110	10100	10110	011010
10011	00111	11111	00001	10000	00000	10000	11111	11100	110011
00110	01101	01010	01011	00101	01011	10101	10101	01001	100110
11111	00110	01100	11001	11111	11000	00100	00000	01100	001111
01010	01100	11001	10011	01010	10010	11101	01010	11001	011010
10011	11001	11111	00110	00011	11110	00001	00000	01111	000011
00110	10011	01010	01100	10110	10100	10111	01010	11010	010110
11100	11110	00011	11001	10000	11111	10000	01000	01111	110000
01001	10100	10110	10011	00101	10101	00101	11010	11010	100101
10011	00111	11100	11110	01100	00111	10000	00010	00011	111100
00110	01101	01001	10100	11001	01101	00101	01110	10110	101001
11100	11001	10011	00111	11111	00001	10000	00000	10000	111111
01001	10011	00110	01101	01010	01011	00101	01011	10101	101010
11100	00111	11111	00110	01100	11001	11111	11000	00100	000000
01001	01101	01010	01100	11001	10011	01010	10010	11101	010101
11111	00001	10011	11001	11111	00110	00011	11110	00001	000000
01010	01011	00110	10011	01010	01100	10110	10100	10111	010101
11111	11000	01100	11110	00011	11001	10000	11111	10000	010000
01010	10010	11001	10100	10110	10011	00101	10101	00101	110101
10011	11110	00011	00111	11100	11110	01100	00111	10000	000100
00110	10100	10110	01101	01001	10100	11001	01101	00101	011101
10000	11111	11100	11001	10011	00111	11111	00001	10000	000001
00101	10101	01001	10011	00110	01101	01010	01011	00101	010111

00000	00001	11111	11112	22222	00000	00001	11111	11112	22222
12345	67890	12345	67890	12345	12345	67890	12345	67890	12345
00000	00000	00000	00000	00000	00000	00000	00000	00000	00000
00001	12121	21200	01111	11222	00001	10101	01000	01111	11000
00002	[illegible]	12100	02222	22111	00000	01010	10100	00000	00111
01120	[illegible]	12211	10001	11111	01100	00111	10011	10001	11111
01121	12202	00111	11112	22000	01101	10000	00111	11110	00000
01122	21020	21011	12220	00222	01100	01000	01011	10000	00000
02210	00222	21122	20002	22222	00010	00000	01100	00000	00000
02211	12010	12022	21110	00111	00011	10010	10000	01110	00111
02212	21101	00222	22221	11000	00010	01101	00000	00001	11000
10110	11001	11112	01201	20120	10110	11001	11110	01001	00100
10111	20122	02012	02012	01012	10111	00100	00010	00010	01010
10112	02210	20212	00120	12201	10110	00010	00010	00100	10001
11200	11112	20020	11202	01201	11000	11110	00000	11000	01001
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13201	32213	31200	222200	11201	12211	11200	222200
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21313	02011	11003	210321	21111	02011	11001	210121
21302	13000	33333	122130	21102	11000	11111	122110
20220	20033	11223	302112	20220	20011	11221	102112
20231	31022	33113	030303	20211	11022	11111	010101
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23102	31110	10012	312013	21102	11110	10012	112011
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21320	31202	02021	233210	21120	11202	02021	211210
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1347036258	1347036250	1346036250	1345035240	1134014230	1132012230
1456372804	1456372004	1456362004	1445352004	1334142003	1332122003
1538624570	1530624570	1530624560	1430524450	1310423340	1310223320
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1783507642	1703507642	1603506642	1503405542	1401304432	1201302232
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2685063174	2605063174	2605063164	2504053154	2403041143	2203021123
2764318750	2764310750	266431[illegible]50	2554310540	2443110430	2223110230
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3827165840	3027165040	3026165040	3025154040	1024143030	1022123030
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4153786120	4153706120	4153606120	4143505120	3131404120	3131202120
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4480615237	4400615237	4400615236	4400514235	3300413214	3300213212
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3606417528	3606417520	3606416520	3505415420	1404314320	1202312320
3718750264	3710750264	3610650264	3510540254	1410430243	1210230223
3827165840	3027165040	3026165040	3025154040	1024143030	1022123030
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4480615237	4400615237	4400615236	4400514235	3300413214	3300213212
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6287354021	6207354021	6206354021	5205344021	4204133021	2202133021
6303825714	6303025714	6303025614	5303024514	4101023413	2101023213
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1

00000 12345	00001 67890	11111 12345	11112 67890	22222 12345	222_3 67890	33333 12345	33334 67890
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